

OLEINIK TYPE ESTIMATES FOR THE OSTROVSKY–HUNTER EQUATION

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ABSTRACT. The Ostrovsky–Hunter equation provides a model for small-amplitude long waves in a rotating fluid of finite depth. It is a nonlinear evolution equation. In this paper we study the well-posedness for the Cauchy problem associated to this equation within a class of bounded discontinuous solutions. We show that we can replace the Kruzkov-type entropy inequalities by an Oleinik-type estimate and prove uniqueness via a nonlocal adjoint problem. An implication is that a shock wave in an entropy weak solution to the Ostrovsky–Hunter equation is admissible only if it jumps down in value (like the inviscid Burgers equation).

1. INTRODUCTION

Our aim is to investigate the well-posedness in classes of discontinuous functions for the equation

$$(1.1) \quad \partial_x(\partial_t u + \partial_x f(u)) = \gamma u, \quad t > 0, x \in \mathbb{R}.$$

We are interested in the Cauchy problem for this equation, so we augment (1.1) with the initial condition

$$(1.2) \quad u(0, x) = u_0(x), \quad x \in \mathbb{R},$$

on which we assume that

$$(1.3) \quad u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x) dx = 0.$$

On the function

$$(1.4) \quad P_0(x) = \int_{-\infty}^x u_0(y) dy, \quad x \in \mathbb{R},$$

we assume that

$$(1.5) \quad \begin{aligned} \|P_0\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \left(\int_{-\infty}^x u_0(y) dy \right)^2 dx < \infty, \\ \int_{\mathbb{R}} P_0(x) dx &= \int_{\mathbb{R}} \left(\int_{-\infty}^x u_0(y) dy \right) dx = 0. \end{aligned}$$

The flux f is assumed to be smooth, Lipschitz continuous and *strictly convex*, namely:

$$(1.6) \quad f \in C^2(\mathbb{R}), \quad f'' \geq C_0, \quad |f'(u)| \leq C_0|u|, \quad u \in \mathbb{R},$$

for some a positive constant C_0 .

The equation (1.1) is the limit of no high-frequency dispersion ($\beta = 0$) of the non-linear evolution equation

$$(1.7) \quad \partial_x(\partial_t u + u\partial_x u - \beta\partial_{xxx}^3 u) = \gamma u,$$

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that was derived by Ostrovsky [24] to model small-amplitude long waves in a rotating fluid of a finite depth. It generalizes the Korteweg-deVries equation (that corresponds to $\gamma = 0$) by the additional term induced by the Coriolis force. Mathematical properties of the Ostrovsky equation (1.7) were studied recently in many details including the local and global well-posedness in energy space [8, 10, 16, 18, 19, 31], stability of solitary waves [14, 17, 20], convergence of solutions in the limit, $\gamma \rightarrow 0$, of the Korteweg-deVries equation [15, 20], and convergence of solutions in the limit, $\beta \rightarrow 0$, of no high-frequency dispersion [4]. (1.1) is deduced considering two asymptotic expansions of the shallow water equations, first with respect to the rotation frequency and then with respect to the amplitude of the waves (see [9, 12]). It is known under different names such as the reduced Ostrovsky equation [25, 28], the Ostrovsky-Hunter equation [1], the short-wave equation [11], and the Vakhnenko equation [21, 26].

Integrating (1.1) on $(-\infty, x)$ we gain the integro-differential formulation of problem (1.1), and (1.2) (see [19])

$$(1.8) \quad \begin{cases} \partial_t u + u \partial_x u = \gamma \int_{-\infty}^x u(t, y) dy, & t > 0, \ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

that is equivalent to

$$(1.9) \quad \begin{cases} \partial_t u + u \partial_x u = \gamma P, & t > 0, \ x \in \mathbb{R}, \\ \partial_x P = u, & t > 0, \ x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

One of the main issues in the analysis of (1.1) is that the equation is not preserving the L^1 norm, the unique useful conserved quantities are

$$t \mapsto \int_{\mathbb{R}} u(t, x) dx, \quad t \mapsto \int_{\mathbb{R}} u^2(t, x) dx.$$

As a consequence the nonlocal source term P and the solution u are a priori only locally bounded and not summable with respect to x . A complete analysis of the well-posedness in that framework can be found in [5, 9] under the additional condition

$$P(t, 0) = 0,$$

that is natural in the reformulation of the boundary value problems for (1.1), see [2, 3]. The equation analyzed in [5, 9] is

$$(1.10) \quad \partial_t u + u \partial_x u = \gamma \int_0^x u(t, y) dy, \quad t > 0, \ x \in \mathbb{R}$$

and not the one in (1.8). The two reformulations (1.8) and (1.10) of (1.1) are not equivalent. Therefore, the well-posedness result of [5, 9] do not apply also to (1.8). Finally, the Kruzkov doubling of variables works for (1.10) but does not for (1.8).

We are interested in the bounded solutions of (1.1) (the ones of [5, 9] are only locally bounded). Indeed, we have (1.5), that is an assumption on the decay at infinity of the initial condition u_0 . The subquadratic assumption (1.6) together with (1.5) guarantees the boundedness of the solutions. Moreover, the convexity of the flux f is necessary for two reasons. It allows us to use a compensated compactness argument for the existence of weak solutions. In addition it gives an Oleinik type estimate. We will show that we can replace the Kruzkov-type entropy inequalities used in [5, 9] by an Oleinik-type estimate and to prove uniqueness via a nonlocal adjoint problem. An implication is that a shock wave in an entropy weak solution to the Ostrovsky-Hunter equation is admissible only if it jumps down in value (like the inviscid Burgers equation).

Definition 1.1. We say that $u \in L^\infty((0, T) \times \mathbb{R})$ is an entropy solution of the initial value problem (1.1), and (1.2) if

- i) u is a distributional solution of (1.8) or equivalently of (1.9);
- ii) for every convex function $\eta \in C^2(\mathbb{R})$ the entropy inequality

$$(1.11) \quad \partial_t \eta(u) + \partial_x q(u) - \gamma \eta'(u) P \leq 0, \quad q(u) = \int^u f'(\xi) \eta'(\xi) d\xi,$$

holds in the sense of distributions in $(0, \infty) \times \mathbb{R}$.

The main result of this paper is the following theorem.

Theorem 1.1. Assume (1.3), (1.4), (1.5) and (1.6). The initial value problem (1.1) and (1.2), possesses an unique entropy solution u in the sense of Definition 1.1.

Moreover, the following statements are equivalent:

- i) u is an entropy solution of (1.8) or (1.9) in the sense of Definition (1.1);
- ii) u is a distributional solution of (1.8) or (1.9) such that for every $T > 0$, there exists $C(T) > 0$ such that

$$(1.12) \quad \frac{u(t, x) - u(t, y)}{x - y} \leq C(T) \left(\frac{1}{t} + 1 \right),$$

for every $0 < t < T$, $x \neq y$.

The paper is organized in three sections. In Section 2, we prove the wellposedness of the approximate solutions of (1.8), or (1.9). In Section 3, we prove the existence of the entropy solutions for (1.8), or (1.9), while in Section 4, we prove an Oleinik type estimate and Theorem 1.1.

2. WELLPOSEDNESS OF THE APPROXIMATE PROBLEM

To prove the existence of entropy solution for (1.8), or (1.9), we analyze the following mixed problem

$$(2.1) \quad \begin{cases} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \gamma P_\varepsilon + \varepsilon \partial_{xx}^2 u_\varepsilon, & t > 0, x \in \mathbb{R}, \\ \partial_x P_\varepsilon = u_\varepsilon, & t > 0, x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases}$$

where $\varepsilon > 0$ is a small fixed number. Clearly, (2.1) is equivalent to the integro-differential problem

$$(2.2) \quad \begin{cases} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon) = \gamma \int_{-\infty}^x u_\varepsilon(t, y) dy + \varepsilon \partial_{xx}^2 u_\varepsilon, & t > 0, x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}. \end{cases}$$

This section is devoted to the wellposedness of (2.1), or (2.2). We assume that

$$(2.3) \quad u_{\varepsilon,0} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R}), \quad \int_{\mathbb{R}} u_{\varepsilon,0}(x) dx = 0.$$

while on the function

$$(2.4) \quad P_{\varepsilon,0}(x) = \int_{-\infty}^x u_{\varepsilon,0}(y) dy, \quad x \in \mathbb{R},$$

we assume that

$$(2.5) \quad \begin{aligned} \|P_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \left(\int_{-\infty}^x u_{\varepsilon,0}(y) dy \right)^2 dx < \infty, \\ \int_{\mathbb{R}} P_{\varepsilon,0}(x) dx &= \int_{\mathbb{R}} \left(\int_{-\infty}^x u_{\varepsilon,0}(y) dy \right) dx = 0. \end{aligned}$$

Fix $0 < \delta < 1$, and let $u_{\varepsilon,\delta} = u_{\varepsilon,\delta}(t, x)$ be the unique classical solution of the following mixed problem [6]:

$$(2.6) \quad \begin{cases} \partial_t u_{\varepsilon,\delta} + \partial_x f(u_{\varepsilon,\delta}) = \gamma P_{\varepsilon,\delta} + \varepsilon \partial_{xx}^2 u_{\varepsilon,\delta}, & t > 0, \ x \in \mathbb{R}, \\ -\delta \partial_{xx}^2 P_{\varepsilon,\delta} + \partial_x P_{\varepsilon,\delta} = u_{\varepsilon,\delta}, & t > 0, \ x \in \mathbb{R}, \\ u_{\varepsilon,\delta}(0, x) = u_{\varepsilon,\delta,0}(x), & x \in \mathbb{R}, \end{cases}$$

where $u_{\varepsilon,\delta,0}$ is a C^∞ approximation of $u_{\varepsilon,0}$ such that

$$(2.7) \quad \begin{aligned} \|u_{\varepsilon,\delta,0}\|_{L^2(\mathbb{R})} &\leq \|u_{\varepsilon,0}\|_{L^2(\mathbb{R})}, \quad \|u_{\varepsilon,\delta,0}\|_{L^\infty(\mathbb{R})} \leq \|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})}, \\ \varepsilon \|\partial_x u_{\varepsilon,\delta,0}\|_{L^2(\mathbb{R})} &\leq C_0, \quad \varepsilon \|\partial_{xx}^2 u_{\varepsilon,\delta,0}\|_{L^2(\mathbb{R})} \leq C_0 \\ \|P_{\varepsilon,\delta,0}\|_{L^2(\mathbb{R})} &\leq \|P_{\varepsilon,0}\|_{L^2(\mathbb{R})}, \quad \delta \|\partial_x P_{\varepsilon,\delta,0}\|_{L^2(\mathbb{R})} \leq C_0, \end{aligned}$$

and C_0 is a constant independent on δ , but dependent on ε .

The main result of this section is the following theorem:

Theorem 2.1. *Let $T > 0$. Assume (1.6), (2.3), (2.5) and (2.4). Then there exist*

$$(2.8) \quad \begin{aligned} u_\varepsilon &\in L^\infty((0, T) \times \mathbb{R}) \cap C((0, T); H^\ell(\mathbb{R})), \quad \ell > 2, \\ P_\varepsilon &\in L^\infty((0, T) \times \mathbb{R}) \cap L^2((0, T) \times \mathbb{R}), \end{aligned}$$

where u_ε is a unique classic solution of the Cauchy problem of (2.1).

Moreover, if u_ε and v_ε are two solutions of (2.1), the following inequality holds

$$(2.9) \quad \|u_\varepsilon(t, \cdot) - v_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq e^{C_\varepsilon(T)t} \|u_{\varepsilon,0} - v_{\varepsilon,0}\|_{L^2(\mathbb{R})},$$

for some suitable $C_\varepsilon(T) > 0$, and every $0 \leq t \leq T$.

We begin by proving some a priori estimates on $u_{\varepsilon,\delta}$ and $P_{\varepsilon,\delta}$, denoting with C_0 the constants which depend on the initial data, and $C(T)$ the constants which depend also on T .

Lemma 2.1. *For each $t \in (0, \infty)$,*

$$(2.10) \quad P_{\varepsilon,\delta}(t, \infty) = \partial_x P_{\varepsilon,\delta}(t, -\infty) = \partial_x P_\varepsilon(t, \infty) = 0.$$

Moreover,

$$(2.11) \quad \delta^2 \|\partial_{xx}^2 P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 = \|u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2.$$

Proof. We begin by proving that (2.10) holds true.

Differentiating the first equation of (2.6) with respect to x , we have

$$(2.12) \quad \partial_x(\partial_t u_{\varepsilon,\delta} + \partial_x f(u_{\varepsilon,\delta}) - \varepsilon \partial_{xx}^2 u_{\varepsilon,\delta}) = \gamma \partial_x P_{\varepsilon,\delta}.$$

For the the smoothness of $u_{\varepsilon,\delta}$, it follows from (2.6) and (2.12) that

$$\begin{aligned} \lim_{x \rightarrow \infty} (\partial_t u_{\varepsilon,\delta} + \partial_x f(u_{\varepsilon,\delta}) - \varepsilon \partial_{xx}^2 u_{\varepsilon,\delta}) &= \gamma P_{\varepsilon,\delta}(t, \infty) = 0, \\ \lim_{x \rightarrow -\infty} \partial_x(\partial_t u_{\varepsilon,\delta} + \partial_x f(u_{\varepsilon,\delta}) - \varepsilon \partial_{xx}^2 u_{\varepsilon,\delta}) &= \gamma \partial_x P_{\varepsilon,\delta}(t, -\infty) = 0, \\ \lim_{x \rightarrow \infty} \partial_x(\partial_t u_{\varepsilon,\delta} + \partial_x f(u_{\varepsilon,\delta}) - \varepsilon \partial_{xx}^2 u_{\varepsilon,\delta}) &= \gamma \partial_x P_{\varepsilon,\delta}(t, \infty) = 0, \end{aligned}$$

which gives (2.10).

Let us show that (2.11) holds true. Squaring the equation for P_ε in (2.6), we get

$$\delta^2 (\partial_{xx}^2 P_{\varepsilon,\delta})^2 + (\partial_x P_{\varepsilon,\delta})^2 - \delta \partial_x ((\partial_x P_{\varepsilon,\delta})^2) = u_{\varepsilon,\delta}^2.$$

Therefore, (2.11) follows from (2.10) and an integration on \mathbb{R} . \square

Lemma 2.2. *For each $t \in (0, \infty)$,*

$$(2.13) \quad \sqrt{\delta} \|\partial_x P_{\varepsilon, \delta}(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_{\varepsilon, \delta}(t, \cdot)\|_{L^2(\mathbb{R})},$$

$$(2.14) \quad \int_{\mathbb{R}} u_{\varepsilon, \delta}(t, x) P_{\varepsilon, \delta}(t, x) dx \leq \|u_{\varepsilon, \delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2.$$

Proof. We begin by proving that (2.13) holds true. Observe that

$$0 \leq (-\delta \partial_{xx}^2 P_\varepsilon + \partial_x P_\varepsilon)^2 = \delta^2 (\partial_{xx}^2 P_\varepsilon)^2 + (\partial_x P_\varepsilon)^2 - \delta \partial_x ((\partial_x P_\varepsilon)^2),$$

that is,

$$(2.15) \quad \delta \partial_x ((\partial_x P_{\varepsilon, \delta})^2) \leq \delta^2 (\partial_{xx}^2 P_{\varepsilon, \delta})^2 + (\partial_x P_{\varepsilon, \delta})^2.$$

Integrating (2.15) on $(-\infty, x)$, we have

$$(2.16) \quad \begin{aligned} \delta (\partial_x P_{\varepsilon, \delta})^2 &\leq \delta^2 \int_{-\infty}^x (\partial_{xx}^2 P_{\varepsilon, \delta})^2 dx + \int_{-\infty}^x (\partial_x P_{\varepsilon, \delta})^2 dx \\ &\leq \delta^2 \int_{\mathbb{R}} (\partial_{xx}^2 P_{\varepsilon, \delta})^2 dx + \int_{\mathbb{R}} (\partial_x P_{\varepsilon, \delta})^2 dx. \end{aligned}$$

It follows from (2.11) and (2.16) that

$$\delta (\partial_x P_{\varepsilon, \delta})^2 \leq \delta^2 \int_{\mathbb{R}} (\partial_{xx}^2 P_{\varepsilon, \delta})^2 dx + \int_{\mathbb{R}} (\partial_x P_{\varepsilon, \delta})^2 dx = \|u_{\varepsilon, \delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2.$$

Therefore,

$$\sqrt{\delta} |\partial_x P_{\varepsilon, \delta}(t, x)| \leq \|u_{\varepsilon, \delta}(t, \cdot)\|_{L^2(\mathbb{R})},$$

which gives (2.13).

Finally, we prove (2.14). Multiplying by $P_{\varepsilon, \delta}$ the equation for $P_{\varepsilon, \delta}$ in (2.6), we get

$$-\delta P_{\varepsilon, \delta} \partial_{xx}^2 P_{\varepsilon, \delta} + P_{\varepsilon, \delta} \partial_x P_{\varepsilon, \delta} = u_{\varepsilon, \delta} P_{\varepsilon, \delta}.$$

An integration on \mathbb{R} and (2.10) give

$$\begin{aligned} \int_{\mathbb{R}} u_{\varepsilon, \delta} P_{\varepsilon, \delta} dx &= \frac{1}{2} \int_{\mathbb{R}} \partial_x (P_\varepsilon)^2 dx - \delta \int_{\mathbb{R}} P_{\varepsilon, \delta} \partial_{xx}^2 P_{\varepsilon, \delta} dx \\ &= -\delta \int_{\mathbb{R}} P_{\varepsilon, \delta} \partial_{xx}^2 P_{\varepsilon, \delta} dx = \delta \int_{\mathbb{R}} (\partial_x P_{\varepsilon, \delta})^2 dx, \end{aligned}$$

that is

$$\int_{\mathbb{R}} u_{\varepsilon, \delta} P_{\varepsilon, \delta} dx = \delta \int_{\mathbb{R}} (\partial_x P_{\varepsilon, \delta})^2 dx.$$

Since $0 < \delta < 1$, for (2.11), we have (2.14). □

Lemma 2.3. *For each $t \in (0, \infty)$, the following inequality holds*

$$(2.17) \quad \|u_{\varepsilon, \delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon e^{2\gamma t} \int_0^t e^{-2\gamma s} \|\partial_x u_{\varepsilon, \delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq e^{2\gamma t} \|u_{\varepsilon, 0}\|_{L^2(\mathbb{R})}^2.$$

In particular, we have

$$(2.18) \quad \|\partial_x P_{\varepsilon, \delta}(t, \cdot)\|_{L^2(\mathbb{R})}, \delta \|\partial_{xx}^2 P_{\varepsilon, \delta}(t, \cdot)\|_{L^2(\mathbb{R})}, \sqrt{\delta} \|\partial_x P_{\varepsilon, \delta}(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq e^{\gamma t} \|u_{\varepsilon, 0}\|_{L^2(\mathbb{R})},$$

Proof. Due to (2.6) and (2.14),

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} u_{\varepsilon,\delta}^2 dx &= 2 \int_{\mathbb{R}} u_{\varepsilon,\delta} \partial_t u_{\varepsilon,\delta} dx \\ &= 2\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\delta} \partial_{xx}^2 u_{\varepsilon,\delta} dx - 2 \int_{\mathbb{R}} u_{\varepsilon,\delta} f'(u_{\varepsilon,\delta}) \partial_x u_{\varepsilon,\delta} dx + 2\gamma \int_{\mathbb{R}} u_{\varepsilon,\delta} P_{\varepsilon,\delta} dx \\ &\leq -2\varepsilon \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\delta})^2 dx + 2\gamma \|u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

The Gronwall Lemma and (2.7) give (2.17).

Finally, (2.18) follows from (2.11), (2.13) and (2.17). \square

Lemma 2.4. *For each $t \geq 0$, we have that*

$$(2.19) \quad \int_0^{-\infty} P_{\varepsilon,\delta}(t, x) dx = a_{\varepsilon,\delta}(t),$$

$$(2.20) \quad \int_0^{\infty} P_{\varepsilon,\delta}(t, x) dx = a_{\varepsilon,\delta}(t),$$

where

$$(2.21) \quad a_{\varepsilon,\delta}(t) = \frac{\delta}{\gamma} \partial_{tx}^2 P_{\varepsilon,\delta}(t, 0) - \frac{1}{\gamma} \partial_t P_{\varepsilon,\delta}(t, 0) + \frac{1}{\gamma} f(0) - \frac{1}{\gamma} f(u_{\varepsilon,\delta}(t, 0)) + \frac{\varepsilon}{\gamma} \partial_x u_{\varepsilon,\delta}(t, 0).$$

In particular,

$$(2.22) \quad \int_{\mathbb{R}} P_{\varepsilon,\delta}(t, x) dx = 0, \quad t \geq 0.$$

Proof. We begin by observing that, integrating the second equation of (2.6) on $(0, x)$, we have that

$$(2.23) \quad \int_0^x u_{\varepsilon,\delta}(t, y) dy = P_{\varepsilon,\delta}(t, x) - P_{\varepsilon,\delta}(t, 0) - \delta \partial_x P_{\varepsilon,\delta}(t, x) + \delta \partial_x P_{\varepsilon,\delta}(t, 0).$$

It follows from (2.10) that

$$(2.24) \quad \lim_{x \rightarrow -\infty} \int_0^x u_{\varepsilon,\delta}(t, y) dy = \int_0^{-\infty} u_{\varepsilon,\delta}(t, x) dx = \delta \partial_x P_{\varepsilon,\delta}(t, 0) - P_{\varepsilon,\delta}(t, 0).$$

Differentiating (2.24) with respect to t , we get

$$(2.25) \quad \frac{d}{dt} \int_0^{-\infty} u_{\varepsilon,\delta}(t, x) dx = \int_0^{-\infty} \partial_t u_{\varepsilon,\delta}(t, x) dx = \delta \partial_{tx}^2 P_{\varepsilon,\delta}(t, 0) - \partial_t P_{\varepsilon,\delta}(t, 0).$$

Integrating the first equation (2.1) on $(0, x)$, we obtain that

$$(2.26) \quad \begin{aligned} \int_0^x \partial_t u_{\varepsilon,\delta}(t, y) dy + f(u_{\varepsilon,\delta}(t, x)) - f(u_{\varepsilon,\delta}(t, 0)) \\ - \varepsilon \partial_x u_{\varepsilon,\delta}(t, x) + \varepsilon \partial_x u_{\varepsilon,\delta}(t, 0) = \gamma \int_0^x P_{\varepsilon,\delta}(t, y) dy. \end{aligned}$$

Being $u_{\varepsilon,\delta}$ a smooth solution of (2.1), we get

$$(2.27) \quad \lim_{x \rightarrow -\infty} \left(f(u_{\varepsilon,\delta}(t, x)) - \varepsilon \partial_x u_{\varepsilon,\delta}(t, x) \right) = f(0).$$

Sending $x \rightarrow -\infty$ in (2.26), for (2.25) and (2.27), we have

$$\begin{aligned} \gamma \int_0^{-\infty} P_{\varepsilon,\delta}(t, x) dx &= \delta \partial_{tx}^2 P_{\varepsilon,\delta}(t, 0) - \partial_t P_{\varepsilon,\delta}(t, 0) \\ &\quad + f(0) - f(u_{\varepsilon,\delta}(t, 0)) + \varepsilon \partial_x u_{\varepsilon,\delta}(t, 0), \end{aligned}$$

which gives (2.19).

Let us show that (2.20) holds true. We begin by observing that, for (2.10) and (2.23),

$$\int_0^\infty u_{\varepsilon,\delta}(t, x) dx = \delta \partial_x P_{\varepsilon,\delta}(t, 0) - P_{\varepsilon,\delta}(t, 0).$$

Therefore,

$$(2.28) \quad \lim_{x \rightarrow \infty} \int_0^x \partial_t u_{\varepsilon,\delta}(t, y) dy = \int_0^\infty \partial_t u_{\varepsilon,\delta}(t, x) dx = \delta \partial_{tx}^2 P_{\varepsilon,\delta}(t, 0) - \partial_t P_{\varepsilon,\delta}(t, 0).$$

Again by the regularity of $u_{\varepsilon,\delta}$,

$$(2.29) \quad \lim_{x \rightarrow \infty} \left(f(u_{\varepsilon,\delta}(t, x)) - \varepsilon \partial_x u_{\varepsilon,\delta}(t, x) \right) = f(0).$$

It follows from (2.26), (2.28) and (2.29) that

$$\begin{aligned} \gamma \int_0^\infty P_{\varepsilon,\delta}(t, x) dx &= \delta \partial_{tx}^2 P_{\varepsilon,\delta}(t, 0) - \partial_t P_{\varepsilon,\delta}(t, 0) \\ &\quad + f(0) - f(u_{\varepsilon,\delta}(t, 0)) + \varepsilon \partial_x u_{\varepsilon,\delta}(t, 0), \end{aligned}$$

which gives (2.20).

Finally, we prove (2.22). It follows from (2.19) that

$$\int_{-\infty}^0 P_{\varepsilon,\delta}(t, x) dx = -a_{\varepsilon,\delta}(t).$$

Therefore, for (2.20),

$$\int_{-\infty}^0 P_{\varepsilon,\delta}(t, x) dx + \int_0^\infty P_{\varepsilon,\delta}(t, x) dx = \int_{\mathbb{R}} P_{\varepsilon,\delta}(t, x) dx = -a_{\varepsilon,\delta}(t) + a_{\varepsilon,\delta}(t) = 0,$$

that is (2.22). \square

Lemma 2.4 says that $P_{\varepsilon,\delta}(t, x)$ is integrable at $\pm\infty$. Therefore, for each $t \geq 0$, we can consider the following function

$$(2.30) \quad F_{\varepsilon,\delta}(t, x) = \int_{-\infty}^x P_{\varepsilon,\delta}(t, y) dy.$$

Lemma 2.5. *Let $T > 0$. There exists a function $C(T) > 0$, independent on δ , such that*

$$(2.31) \quad \|P_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})} \leq C(T),$$

$$(2.32) \quad \|P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T),$$

$$(2.33) \quad \delta \|\partial_x P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T),$$

where

$$(2.34) \quad I_{T,1} = (0, T) \times \mathbb{R}.$$

In particular, we have

$$(2.35) \quad \delta \left| \int_0^t \int_{\mathbb{R}} P_{\varepsilon,\delta} \partial_{tx}^2 P_{\varepsilon,\delta} ds dx \right| \leq C(T), \quad 0 < t < T.$$

Proof. Integrating the second equation of (2.1) on $(-\infty, x)$, for (2.10), we have that

$$(2.36) \quad \int_{-\infty}^x u_{\varepsilon,\delta}(t, y) dy = P_{\varepsilon,\delta}(t, x) - \delta \partial_x P_{\varepsilon,\delta}(t, x).$$

Differentiating (2.36) with respect to t , we get

$$(2.37) \quad \frac{d}{dt} \int_{-\infty}^x u_{\varepsilon,\delta}(t, y) dy = \int_{-\infty}^x \partial_t u_{\varepsilon,\delta}(t, y) dy = \partial_t P_{\varepsilon,\delta}(t, x) - \delta \partial_{tx}^2 P_{\varepsilon,\delta}(t, x).$$

It follows from an integration of the first equation of (2.6) on $(-\infty, x)$ and (2.30) that

$$(2.38) \quad \int_{-\infty}^x \partial_t u_{\varepsilon,\delta}(t, y) dy + f(u_{\varepsilon,\delta}(t, x)) - \varepsilon \partial_x u_{\varepsilon,\delta}(t, x) = \gamma F_{\varepsilon,\delta}(t, x).$$

Due to (2.37) and (2.38), we have

$$(2.39) \quad \partial_t P_{\varepsilon,\delta}(t, x) - \delta \partial_{tx}^2 P_{\varepsilon,\delta}(t, x) = \gamma F_{\varepsilon,\delta}(t, x) - f(u_{\varepsilon,\delta}(t, x)) + \varepsilon \partial_x u_{\varepsilon,\delta}(t, x).$$

Multiplying (2.39) by $P_{\varepsilon,\delta} - \delta \partial_x P_{\varepsilon,\delta}$, we have

$$(2.40) \quad \begin{aligned} (\partial_t P_{\varepsilon,\delta} - \delta \partial_{tx}^2 P_{\varepsilon,\delta})(P_{\varepsilon,\delta} - \delta \partial_x P_{\varepsilon,\delta}) = & \gamma F_{\varepsilon,\delta}(P_{\varepsilon,\delta} - \delta \partial_x P_{\varepsilon,\delta}) \\ & - f(u_{\varepsilon,\delta})(P_{\varepsilon,\delta} - \delta \partial_x P_{\varepsilon,\delta}) \\ & + \varepsilon \partial_x u_{\varepsilon,\delta}(P_{\varepsilon,\delta} - \delta \partial_x P_{\varepsilon,\delta}). \end{aligned}$$

Integrating (2.40) on $(0, x)$, we have

$$(2.41) \quad \begin{aligned} & \int_0^x \partial_t P_{\varepsilon,\delta} P_{\varepsilon,\delta} dy - \delta \int_0^x \partial_t P_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy \\ & \quad - \delta \int_0^x P_{\varepsilon,\delta} \partial_{tx}^2 P_{\varepsilon,\delta} dy + \delta^2 \int_0^x \partial_{tx}^2 P_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy \\ & = \gamma \int_0^x F_{\varepsilon,\delta} P_{\varepsilon,\delta} dy - \gamma \delta \int_0^x F_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy \\ & \quad - \int_0^x f(u_{\varepsilon,\delta}) P_{\varepsilon,\delta} dy + \delta \int_0^x f(u_{\varepsilon,\delta}) \partial_x P_{\varepsilon,\delta} dy \\ & \quad + \varepsilon \int_0^x \partial_x u_{\varepsilon,\delta} P_{\varepsilon,\delta} dy - \varepsilon \delta \int_0^x \partial_x u_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy. \end{aligned}$$

We observe that

$$(2.42) \quad -\delta \int_0^x \partial_x P_{\varepsilon,\delta} \partial_t P_{\varepsilon,\delta} dy = -\delta P_{\varepsilon,\delta} \partial_t P_{\varepsilon,\delta} + \delta P_{\varepsilon,\delta}(t, 0) \partial_t P_{\varepsilon,\delta}(t, 0) + \delta \int_0^x P_{\varepsilon,\delta} \partial_{tx}^2 P_{\varepsilon,\delta} dy.$$

Therefore, (2.41) and (2.42) give

$$(2.43) \quad \begin{aligned} & \int_0^x \partial_t P_{\varepsilon,\delta} P_{\varepsilon,\delta} dy + \delta^2 \int_0^x \partial_{tx}^2 P_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy \\ & = \delta P_{\varepsilon,\delta} \partial_t P_{\varepsilon,\delta} - \delta P_{\varepsilon,\delta}(t, 0) \partial_t P_{\varepsilon,\delta}(t, 0) + \gamma \int_0^x F_{\varepsilon,\delta} P_{\varepsilon,\delta} dy \\ & \quad - \gamma \delta \int_0^x F_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy - \int_0^x f(u_{\varepsilon,\delta}) P_{\varepsilon,\delta} dy + \delta \int_0^x f(u_{\varepsilon,\delta}) \partial_x P_{\varepsilon,\delta} dy \\ & \quad + \varepsilon \int_0^x \partial_x u_{\varepsilon,\delta} P_{\varepsilon,\delta} dy - \varepsilon \delta \int_0^x \partial_x u_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy. \end{aligned}$$

Sending $x \rightarrow -\infty$, for (2.10), we get

$$\begin{aligned}
 (2.44) \quad & \int_0^{-\infty} \partial_t P_{\varepsilon,\delta} P_{\varepsilon,\delta} dy + \delta^2 \int_0^{-\infty} \partial_{tx}^2 P_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy \\
 &= -\delta P_{\varepsilon,\delta}(t, 0) \partial_t P_{\varepsilon,\delta}(t, 0) + \gamma \int_0^{-\infty} F_{\varepsilon,\delta} P_{\varepsilon,\delta} dy \\
 &\quad - \gamma \delta \int_0^{-\infty} F_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy - \int_0^{-\infty} f(u_{\varepsilon,\delta}) P_{\varepsilon,\delta} dy \\
 &\quad + \delta \int_0^{-\infty} f(u_{\varepsilon,\delta}) \partial_x P_{\varepsilon,\delta} dy + \varepsilon \int_0^{-\infty} \partial_x u_{\varepsilon,\delta} P_{\varepsilon,\delta} dy \\
 &\quad - \varepsilon \delta \int_0^{-\infty} \partial_x u_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy,
 \end{aligned}$$

while sending $x \rightarrow \infty$,

$$\begin{aligned}
 (2.45) \quad & \int_0^{\infty} \partial_t P_{\varepsilon,\delta} P_{\varepsilon,\delta} dy + \delta^2 \int_0^{\infty} \partial_{tx}^2 P_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy \\
 &= -\delta P_{\varepsilon,\delta}(t, 0) \partial_t P_{\varepsilon,\delta}(t, 0) + \gamma \int_0^{\infty} F_{\varepsilon,\delta} P_{\varepsilon,\delta} dy - \gamma \delta \int_0^{\infty} F_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy \\
 &\quad - \int_0^{\infty} f(u_{\varepsilon,\delta}) P_{\varepsilon,\delta} dy + \delta \int_0^{\infty} f(u_{\varepsilon,\delta}) \partial_x P_{\varepsilon,\delta} dy \\
 &\quad + \varepsilon \int_0^{\infty} \partial_x u_{\varepsilon,\delta} P_{\varepsilon,\delta} dy - \varepsilon \delta \int_0^{\infty} \partial_x u_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dy.
 \end{aligned}$$

Since

$$\begin{aligned}
 \int_{\mathbb{R}} P_{\varepsilon,\delta} \partial_t P_{\varepsilon,\delta} dx &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} P_{\varepsilon,\delta}^2 dx, \\
 \delta^2 \int_{\mathbb{R}} \partial_{tx}^2 P_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dx &= \frac{\delta^2}{2} \frac{d}{dt} \int_{\mathbb{R}} (\partial_x P_{\varepsilon,\delta})^2 dx,
 \end{aligned}$$

it follows from (2.44) and (2.45) that

$$\begin{aligned}
 (2.46) \quad & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} P_{\varepsilon,\delta}^2 dx + \frac{\delta^2}{2} \frac{d}{dt} \int_{\mathbb{R}} (\partial_x P_{\varepsilon,\delta})^2 dx \\
 &= \gamma \int_{\mathbb{R}} F_{\varepsilon,\delta} P_{\varepsilon,\delta} dx - \gamma \delta \int_{\mathbb{R}} F_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dx \\
 &\quad - \int_{\mathbb{R}} f(u_{\varepsilon,\delta}) P_{\varepsilon,\delta} dx + \delta \int_{\mathbb{R}} f(u_{\varepsilon,\delta}) \partial_x P_{\varepsilon,\delta} dx \\
 &\quad + \varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon,\delta} P_{\varepsilon,\delta} dx - \varepsilon \delta \int_{\mathbb{R}} \partial_x u_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dx.
 \end{aligned}$$

Due to (2.22) and (2.30),

$$\begin{aligned}
 (2.47) \quad & 2\gamma \int_{\mathbb{R}} F_{\varepsilon,\delta} P_{\varepsilon,\delta} dx = 2\gamma \int_{\mathbb{R}} F_{\varepsilon,\delta} \partial_x F_{\varepsilon,\delta} dx = \gamma (F_{\varepsilon,\delta}(t, \infty))^2 \\
 &= \gamma \left(\int_{\mathbb{R}} P_{\varepsilon,\delta}(t, x) dx \right)^2 = 0.
 \end{aligned}$$

(2.46) and (2.47) give

$$\begin{aligned}
(2.48) \quad & \frac{d}{dt} \left(\int_{\mathbb{R}} P_{\varepsilon,\delta}^2 dx + \delta^2 \int_{\mathbb{R}} (\partial_x P_{\varepsilon,\delta})^2 dx \right) \\
&= -2\gamma\delta \int_{\mathbb{R}} F_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dx - 2 \int_{\mathbb{R}} f(u_{\varepsilon,\delta}) P_{\varepsilon,\delta} dx \\
&\quad + 2\delta \int_{\mathbb{R}} f(u_{\varepsilon,\delta}) \partial_x P_{\varepsilon,\delta} dx + 2\varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon,\delta} P_{\varepsilon,\delta} dx \\
&\quad - 2\varepsilon\delta \int_{\mathbb{R}} \partial_x u_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dx.
\end{aligned}$$

Thanks to (2.10), (2.22) and (2.30),

$$(2.49) \quad -2\delta\gamma \int_{\mathbb{R}} \partial_x P_{\varepsilon,\delta} F_{\varepsilon,\delta} dx = 2\delta\gamma \int_{\mathbb{R}} P_{\varepsilon,\delta} \partial_x F_{\varepsilon,\delta} dx = 2\delta\gamma \int_{\mathbb{R}} P_{\varepsilon,\delta}^2 dx \leq 2\gamma \int_{\mathbb{R}} P_{\varepsilon,\delta}^2 dx,$$

while for (2.10),

$$(2.50) \quad 2\varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon,\delta} P_{\varepsilon,\delta} dx = -2\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dx.$$

Hence, for (1.6), (2.49) and (2.50), we get

$$\begin{aligned}
& \frac{d}{dt} \left(\int_{\mathbb{R}} P_{\varepsilon,\delta}^2 dx + \delta^2 \int_{\mathbb{R}} (\partial_x P_{\varepsilon,\delta})^2 dx \right) \\
& \leq 2\gamma \int_{\mathbb{R}} P_{\varepsilon,\delta}^2 dx - 2 \int_{\mathbb{R}} f(u_{\varepsilon,\delta}) P_{\varepsilon,\delta} dx + 2\delta \int_{\mathbb{R}} f(u_{\varepsilon,\delta}) \partial_x P_{\varepsilon,\delta} dx \\
& \quad - 2\varepsilon \int_{\mathbb{R}} u_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dx - 2\varepsilon\delta \int_{\mathbb{R}} \partial_x u_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dx \\
& \leq 2\gamma \int_{\mathbb{R}} P_{\varepsilon,\delta}^2 dx + 2 \left| \int_{\mathbb{R}} f(u_{\varepsilon,\delta}) P_{\varepsilon,\delta} dx \right| + 2\delta \left| \int_{\mathbb{R}} f(u_{\varepsilon,\delta}) \partial_x P_{\varepsilon,\delta} dx \right| \\
& \quad + 2\varepsilon \left| \int_{\mathbb{R}} u_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dx \right| + 2\varepsilon\delta \left| \int_{\mathbb{R}} \partial_x u_{\varepsilon,\delta} \partial_x P_{\varepsilon,\delta} dx \right| \\
& \leq 2\gamma \int_{\mathbb{R}} P_{\varepsilon,\delta}^2 dx + 2 \int_{\mathbb{R}} |f(u_{\varepsilon,\delta})| |P_{\varepsilon,\delta}| dx + 2\delta \int_{\mathbb{R}} |f(u_{\varepsilon,\delta})| |\partial_x P_{\varepsilon,\delta}| dx \\
& \quad + 2\varepsilon \int_{\mathbb{R}} |u_{\varepsilon,\delta}| |\partial_x P_{\varepsilon,\delta}| dx + 2\varepsilon\delta \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}| |\partial_x P_{\varepsilon,\delta}| dx \\
& \leq 2\gamma \int_{\mathbb{R}} P_{\varepsilon,\delta}^2 dx + 2C_0 \int_{\mathbb{R}} |P_{\varepsilon,\delta}| u_{\varepsilon,\delta}^2 dx + 2C_0\delta \int_{\mathbb{R}} |\partial_x P_{\varepsilon,\delta}| u_{\varepsilon,\delta}^2 dx \\
& \quad + 2\varepsilon \int_{\mathbb{R}} |u_{\varepsilon,\delta}| |\partial_x P_{\varepsilon,\delta}| dx + 2\varepsilon\delta \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}| |\partial_x P_{\varepsilon,\delta}| dx.
\end{aligned}$$

For the Young inequality,

$$\begin{aligned}
& 2\varepsilon \int_{\mathbb{R}} |\partial_x P_{\varepsilon,\delta}| |u_{\varepsilon,\delta}| dx \leq \varepsilon \|\partial_x P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon \|u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
& 2\varepsilon\delta \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}| |\partial_x P_{\varepsilon,\delta}| dx = \int_{\mathbb{R}} \left| \frac{\varepsilon \partial_x u_{\varepsilon,\delta}}{\sqrt{\gamma}} \right| |2\sqrt{\gamma}\delta \partial_x P_{\varepsilon,\delta}| dx \\
& \leq \frac{\varepsilon^2}{2\gamma} \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\delta^2\gamma \|\partial_x P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
(2.51) \quad \frac{d}{dt}G(t) - 2\gamma G(t) &\leq \varepsilon \|u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2C_0 \int_{\mathbb{R}} |P_{\varepsilon,\delta}| u_{\varepsilon,\delta}^2 dx \\
&+ 2C_0 \delta \int_{\mathbb{R}} |\partial_x P_{\varepsilon,\delta}| u_{\varepsilon,\delta}^2 dx + \varepsilon \|\partial_x P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&+ \frac{\varepsilon^2}{2\gamma} \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

where

$$(2.52) \quad G(t) = \|P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \delta^2 \|\partial_x P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2.$$

We observe that, for (2.17),

$$(2.53) \quad 2C_0 \int_{\mathbb{R}} |P_{\varepsilon,\delta}| u_{\varepsilon,\delta}^2 dx \leq C_0 e^{2\gamma t} \|P_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})},$$

where $I_{T,1}$ is defined in (2.34).

Since $0 < \delta < 1$, it follows from (2.17) and (2.18) that

$$\begin{aligned}
(2.54) \quad 2C_0 \delta \int_{\mathbb{R}} |\partial_x P_{\varepsilon,\delta}| u_{\varepsilon,\delta}^2 dx &\leq 2C_0 \delta \|\partial_x P_{\varepsilon,\delta}(t, \cdot)\|_{L^\infty(\mathbb{R})} \|u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq 2\sqrt{\delta} C_0 e^{3\gamma t} \leq C_0 e^{3\gamma t}.
\end{aligned}$$

Again by (2.18), we have that

$$(2.55) \quad \varepsilon \|\partial_x P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \varepsilon \|\partial_x P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C_0 e^{2\gamma t}.$$

Therefore, (2.17), (2.54) and (2.55) give

$$\frac{d}{dt}G(t) - 2\gamma G(t) \leq C_0 \left(\|P_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})} + 1 \right) e^{2\gamma t} + C_0 e^{3\gamma t} + \frac{\varepsilon^2}{2\gamma} \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2.$$

The Gronwall Lemma, (2.7), (2.17) and (2.52) give

$$\begin{aligned}
&\|P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \delta^2 \|\partial_x P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq \|P_{\varepsilon,0}\|_{L^2(0,\infty)}^2 e^{2\gamma t} + \left(\|P_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})} + 1 \right) t e^{2\gamma t} + C_0 t e^{3\gamma t} \\
&\quad + \frac{\varepsilon^2 e^{2\gamma t}}{2\gamma} \int_0^t e^{-2\gamma s} \|\partial_x u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
&\leq \|P_{\varepsilon,0}\|_{L^2(0,\infty)}^2 e^{2\gamma t} + \left(\|P_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})} + 1 \right) t e^{2\gamma t} + C_0 t e^{3\gamma t} + C_0 e^{2\gamma t}.
\end{aligned}$$

Hence,

$$(2.56) \quad \|P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \delta^2 \|\partial_x P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T) \left(\|P_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})} + 1 \right).$$

Due to (2.18), (2.56) and the Hölder inequality,

$$\begin{aligned}
P_{\varepsilon,\delta}^2(t, x) &\leq 2 \int_{\mathbb{R}} |P_{\varepsilon,\delta}| |\partial_x P_{\varepsilon,\delta}| dx \leq 2 \|P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})} \\
&\leq 2 \sqrt{C(T) \left(\|P_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})} + 1 \right)} \sqrt{C_0} e^{\gamma t} \leq C(T) \left(\|P_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})} + 1 \right).
\end{aligned}$$

Therefore,

$$\|P_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})}^2 - C(T) \|P_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})} - C(T) \leq 0,$$

which gives (2.31).

(2.32) and (2.33) follow from (2.31) and (2.56).

Let us show that (2.35) holds true. Multiplying (2.39) by $P_{\varepsilon,\delta}$, an integration on \mathbb{R} and (2.47) give

$$\begin{aligned} 2\delta \int_{\mathbb{R}} \partial_{tx}^2 P_{\varepsilon,\delta} P_{\varepsilon,\delta} dx &= \frac{d}{dt} \|P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2\gamma \int_{\mathbb{R}} F_{\varepsilon,\delta} P_{\varepsilon,\delta} dx \\ &\quad + 2 \int_{\mathbb{R}} f(u_{\varepsilon,\delta}) P_{\varepsilon,\delta} dx - 2\varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon,\delta} P_{\varepsilon,\delta} dx \\ &= \frac{d}{dt} \|P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2 \int_{\mathbb{R}} f(u_{\varepsilon,\delta}) P_{\varepsilon,\delta} dx - 2\varepsilon \int_{\mathbb{R}} \partial_x u_{\varepsilon,\delta} P_{\varepsilon,\delta} dx. \end{aligned}$$

An integration on $(0, t)$ gives

$$\begin{aligned} 2\delta \int_0^t \int_{\mathbb{R}} \partial_{tx}^2 P_{\varepsilon,\delta} P_{\varepsilon,\delta} dx &= \|P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 - \|P_{\varepsilon,\delta,0}\|_{L^2(\mathbb{R})}^2 \\ &\quad + 2 \int_0^t \int_{\mathbb{R}} f(u_{\varepsilon,\delta}) P_{\varepsilon,\delta} dx - 2\varepsilon \int_0^t \int_{\mathbb{R}} \partial_x u_{\varepsilon,\delta} P_{\varepsilon,\delta} dx. \end{aligned}$$

It follows from (1.6), (2.17), (2.31) and (2.32) that

$$\begin{aligned} 2\delta \left| \int_0^t \int_{\mathbb{R}} \partial_{tx}^2 P_{\varepsilon,\delta} P_{\varepsilon,\delta} ds dx \right| &\leq \|P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|P_{\varepsilon,\delta,0}\|_{L^2(\mathbb{R})}^2 \\ &\quad + 2 \int_0^t \int_{\mathbb{R}} |f(u_{\varepsilon,\delta})| |P_{\varepsilon,\delta}| ds dx \\ &\quad + 2\varepsilon \int_0^t \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}| |P_{\varepsilon,\delta}| ds dx \\ &\leq \|P_{\varepsilon,\delta,0}\|_{L^2(\mathbb{R})}^2 + 2C(T) \int_0^t \int_{\mathbb{R}} u_{\varepsilon,\delta}^2 ds dx \\ &\quad + 2\varepsilon \int_0^t \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}| |P_{\varepsilon,\delta}| ds dx + C(T) \\ &\leq \|P_{\varepsilon,\delta,0}\|_{L^2(\mathbb{R})}^2 + C(T) \\ &\quad + 2\varepsilon \int_0^t \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}| |P_{\varepsilon,\delta}| ds dx. \end{aligned}$$

Observe that, thanks to (2.17),

$$\begin{aligned} (2.57) \quad &\varepsilon \int_0^t \|\partial_x u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ &\leq \varepsilon e^{2\gamma t} \int_0^t e^{-2\gamma s} \|\partial_x u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T). \end{aligned}$$

Due to Young inequality,

$$\begin{aligned} (2.58) \quad &2\varepsilon \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}| |P_{\varepsilon,\delta}| ds dx \\ &\leq \varepsilon \|P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq C(T) + \varepsilon \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Then, for (2.57) and (2.58), we have that

$$2\varepsilon \int_0^t \int_{\mathbb{R}} |P_{\varepsilon,\delta}| |\partial_x u_{\varepsilon,\delta}| ds dx$$

$$\leq \varepsilon \int_0^t \|P_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + \varepsilon \int_0^t \|\partial_x u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T).$$

Therefore,

$$2\delta \left| \int_0^t \int_{\mathbb{R}} P_{\varepsilon,\delta} \partial_{tx}^2 P_{\varepsilon,\delta} ds dx \right| \leq \|P_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2 + C(T),$$

which gives (2.35). □

Lemma 2.6. *Let $T > 0$. Then,*

$$(2.59) \quad \|u_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})} \leq \|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} + C(T),$$

where $I_{T,1}$ is defined in (2.34).

Proof. Due to (2.6) and (2.31),

$$\partial_t u_{\varepsilon,\delta} + \partial_x f(u_{\varepsilon,\delta}) - \varepsilon \partial_{xx}^2 u_{\varepsilon,\delta} \leq \gamma C(T).$$

Since the map

$$\mathcal{F}(t) := \|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} + \gamma C(T)t,$$

solves the equation

$$\frac{d\mathcal{F}}{dt} = \gamma C(T)$$

and

$$\max\{u_{\varepsilon,\delta}(0, x), 0\} \leq \mathcal{F}(t), \quad (t, x) \in I_{T,1},$$

the comparison principle for parabolic equations implies that

$$u_{\varepsilon,\delta}(t, x) \leq \mathcal{F}(t), \quad (t, x) \in I_{T,1}.$$

In a similar way we can prove that

$$u_{\varepsilon,\delta}(t, x) \geq -\mathcal{F}(t), \quad (t, x) \in I_{T,1}.$$

Therefore,

$$|u_{\varepsilon,\delta}(t, x)| \leq \|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} + \gamma C(T)t \leq \|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} + C(T),$$

which gives (2.59). □

Lemma 2.7. *Let $T > 0$ and $0 < \delta < 1$. We have that*

$$(2.60) \quad \varepsilon \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \int_0^t \|\partial_{xx}^2 u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T).$$

Proof. Let $0 < t < T$. Multiplying (2.6) by $-\varepsilon \partial_{xx}^2 u_{\varepsilon,\delta}$, we have

$$(2.61) \quad \begin{aligned} -\varepsilon \partial_{xx}^2 u_{\varepsilon,\delta} \partial_t u_{\varepsilon,\delta} + \varepsilon^2 \partial_{xx}^2 u_{\varepsilon,\delta}^2 \\ = -\gamma \varepsilon P_{\varepsilon,\delta} \partial_{xx}^2 u_{\varepsilon,\delta} - \varepsilon f'(u_{\varepsilon,\delta}) \partial_x u_{\varepsilon,\delta} \partial_{xx}^2 u_{\varepsilon,\delta}. \end{aligned}$$

Since

$$-\varepsilon \int_{\mathbb{R}} \partial_{xx}^2 u_{\varepsilon,\delta} \partial_t u_{\varepsilon,\delta} dx = \frac{d}{dt} \left(\frac{\varepsilon}{2} \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\delta})^2 \right),$$

integrating (2.61) on \mathbb{R} , we get

$$\begin{aligned} \frac{d}{dt} \left(\varepsilon \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\delta})^2 dx \right) + 2\varepsilon^2 \int_{\mathbb{R}} (\partial_{xx}^2 u_{\varepsilon,\delta})^2 dx \\ = -2\gamma \varepsilon \int_{\mathbb{R}} P_{\varepsilon,\delta} \partial_{xx}^2 u_{\varepsilon,\delta} dx \\ - 2\varepsilon \int_{\mathbb{R}} f'(u_{\varepsilon,\delta}) \partial_x u_{\varepsilon,\delta} \partial_{xx}^2 u_{\varepsilon,\delta} dx. \end{aligned}$$

Due to (2.17), (2.32), (2.59) and the Young inequality,

$$\begin{aligned}
& -2\gamma\varepsilon \int_{\mathbb{R}} P_{\varepsilon,\delta} \partial_{xx}^2 u_{\varepsilon,\delta} dx \\
& \leq 2\gamma\varepsilon \left| \int_{\mathbb{R}} P_{\varepsilon,\delta} \partial_{xx}^2 u_{\varepsilon,\delta} dx \right| \\
& \leq 2 \int_{\mathbb{R}} \left| \sqrt{2}\gamma P_{\varepsilon,\delta} \right| \left| \frac{\varepsilon \partial_{xx}^2 u_{\varepsilon,\delta}}{\sqrt{2}} \right| dx \\
& \leq 2\gamma^2 \|P_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon^2}{2} \|\partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
& \leq C(T) + \frac{\varepsilon^2}{2} \|\partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
& -2\varepsilon \int_{\mathbb{R}} f'(u_{\varepsilon,\delta}) \partial_x u_{\varepsilon,\delta} \partial_{xx}^2 u_{\varepsilon,\delta} dx \\
& \leq 2\varepsilon \left| \int_{\mathbb{R}} f'(u_{\varepsilon,\delta}) \partial_x u_{\varepsilon,\delta} \partial_{xx}^2 u_{\varepsilon,\delta} dx \right| \\
& \leq 2 \int_{\mathbb{R}} \left| \sqrt{2}f'(u_{\varepsilon,\delta}) \partial_x u_{\varepsilon,\delta} \right| \left| \frac{\varepsilon \partial_{xx}^2 u_{\varepsilon,\delta}}{\sqrt{2}} \right| dx \\
& \leq 2 \int_{\mathbb{R}} (f'(u_{\varepsilon,\delta}))^2 (\partial_x u_{\varepsilon,\delta})^2 + \frac{\varepsilon^2}{2} \int_{\mathbb{R}} (\partial_{xx}^2 u_{\varepsilon,\delta})^2 dx \\
& \leq 2 \|f'\|_{L^\infty(I_{T,2})}^2 \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon^2}{2} \|\partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

where

$$(2.62) \quad I_{T,2} = \left(-\|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} - C(T), \|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} + C(T) \right).$$

Therefore,

$$\begin{aligned}
& \frac{d}{dt} \left(\varepsilon \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + 2\varepsilon^2 \|\partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
& \leq \varepsilon^2 \|\partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|f'\|_{L^\infty(I_{T,2})}^2 \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(T),
\end{aligned}$$

that is

$$\begin{aligned}
& \frac{d}{dt} \left(\varepsilon \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + \varepsilon^2 \|\partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
& \leq \|f'\|_{L^\infty(I_{T,2})}^2 \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(T).
\end{aligned}$$

An integration on $(0, t)$ and (2.7) give

$$\begin{aligned}
(2.63) \quad & \varepsilon \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \int_0^t \|\partial_{xx}^2 u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
& \leq 2 \|f'\|_{L^\infty(I_{T,2})}^2 \int_0^t \|\partial_x u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds + C(T).
\end{aligned}$$

(2.60) follows from (2.57) and (2.63). \square

Lemma 2.8. *Let $T > 0$ and $0 < \delta < 1$. We have that*

$$(2.64) \quad \|\partial_x u_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})} \leq C(T),$$

where $I_{T,1}$ is defined in (2.34). Moreover,

$$(2.65) \quad \varepsilon \left\| \partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \int_0^t \left\| \partial_{xxx}^3 u_{\varepsilon,\delta}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \leq C(T).$$

Proof. Let $0 < t < T$. Multiplying (2.6) by $\varepsilon \partial_{xxxx}^4 u_{\varepsilon,\delta}$, we have

$$(2.66) \quad \begin{aligned} & \varepsilon \partial_{xxxx}^4 u_{\varepsilon,\delta} \partial_t u_{\varepsilon,\delta} - \varepsilon^2 \partial_{xxxx}^4 u_{\varepsilon,\delta} \partial_{xx}^2 u_{\varepsilon,\delta} \\ & = + \varepsilon \gamma P_{\varepsilon,\delta} \partial_{xxxx}^4 u_{\varepsilon,\delta} - \varepsilon f'(u_{\varepsilon,\delta}) \partial_x u_{\varepsilon,\delta} \partial_{xxxx}^4 u_{\varepsilon,\delta}. \end{aligned}$$

Since

$$\begin{aligned} & \varepsilon \int_{\mathbb{R}} \partial_{xxxx}^4 u_{\varepsilon,\delta} \partial_t u_{\varepsilon,\delta} dx = \frac{d}{dt} \left(\frac{\varepsilon}{2} \int_{\mathbb{R}} (\partial_{xx}^2 u_{\varepsilon,\delta})^2 dx \right), \\ & - \varepsilon^2 \int_{\mathbb{R}} \partial_{xxxx}^4 u_{\varepsilon,\delta} \partial_{xx}^2 u_{\varepsilon,\delta} dx = \varepsilon^2 \int_{\mathbb{R}} (\partial_{xxx}^3 u_{\varepsilon,\delta})^2 dx, \\ & \varepsilon \gamma \int_{\mathbb{R}} P_{\varepsilon,\delta} \partial_{xxxx}^4 u_{\varepsilon,\delta} dx = - \varepsilon \gamma \int_{\mathbb{R}} \partial_x P_{\varepsilon,\delta} \partial_{xxx}^3 u_{\varepsilon,\delta} dx, \\ & - \varepsilon \int_{\mathbb{R}} f'(u_{\varepsilon,\delta}) \partial_x u_{\varepsilon,\delta} \partial_{xxxx}^4 u_{\varepsilon,\delta} dx = \varepsilon \int_{\mathbb{R}} f''(u_{\varepsilon,\delta}) (\partial_x u_{\varepsilon,\delta})^2 \partial_{xxx}^3 u_{\varepsilon,\delta} dx \\ & \quad + \varepsilon \int_{\mathbb{R}} f'(u_{\varepsilon,\delta}) \partial_{xx}^2 u_{\varepsilon,\delta} \partial_{xxx}^3 u_{\varepsilon,\delta} dx, \end{aligned}$$

integrating (2.61) on \mathbb{R} , we get

$$\begin{aligned} & \frac{d}{dt} \left(\varepsilon \int_{\mathbb{R}} (\partial_{xx}^2 u_{\varepsilon,\delta})^2 dx \right) + 2\varepsilon^2 \int_{\mathbb{R}} (\partial_{xxx}^3 u_{\varepsilon,\delta})^2 dx \\ & = - 2\varepsilon \gamma \int_{\mathbb{R}} \partial_x P_{\varepsilon,\delta} \partial_{xxx}^3 u_{\varepsilon,\delta} dx \\ & \quad + 2\varepsilon \int_{\mathbb{R}} f''(u_{\varepsilon,\delta}) (\partial_x u_{\varepsilon,\delta})^2 \partial_{xxx}^3 u_{\varepsilon,\delta} dx \\ & \quad + 2\varepsilon \int_{\mathbb{R}} f'(u_{\varepsilon,\delta}) \partial_{xx}^2 u_{\varepsilon,\delta} \partial_{xxx}^3 u_{\varepsilon,\delta} dx. \end{aligned}$$

Due to (2.18), (2.59), (2.60) and the Young inequality,

$$\begin{aligned} & - 2\varepsilon \gamma \int_{\mathbb{R}} \partial_x P_{\varepsilon,\delta} \partial_{xxx}^3 u_{\varepsilon,\delta} dx \\ & \leq 2\varepsilon \gamma \left| \int_{\mathbb{R}} \partial_x P_{\varepsilon,\delta} \partial_{xxx}^3 u_{\varepsilon,\delta} dx \right| \\ & \leq 2 \int_{\mathbb{R}} \left| \sqrt{3} \gamma \partial_x P_{\varepsilon,\delta} \right| \left| \frac{\varepsilon \partial_{xxx}^3 u_{\varepsilon,\delta}}{\sqrt{3}} \right| dx \\ & \leq 3\gamma^2 \left\| \partial_x P_{\varepsilon,\delta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon^2}{3} \left\| \partial_{xxx}^3 u_{\varepsilon,\delta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\ & \leq C(T) + \frac{\varepsilon^2}{3} \left\| \partial_{xxx}^3 u_{\varepsilon,\delta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\ & 2\varepsilon \int_{\mathbb{R}} f''(u_{\varepsilon,\delta}) (\partial_x u_{\varepsilon,\delta})^2 \partial_{xxx}^3 u_{\varepsilon,\delta} dx \\ & \leq 2\varepsilon \left| \int_{\mathbb{R}} f''(u_{\varepsilon,\delta}) (\partial_x u_{\varepsilon,\delta})^2 \partial_{xxx}^3 u_{\varepsilon,\delta} dx \right| \end{aligned}$$

$$\begin{aligned}
&\leq 2 \int_{\mathbb{R}} \left| \sqrt{3} f''(u_{\varepsilon,\delta}) (\partial_x u_{\varepsilon,\delta})^2 \right| \left| \frac{\varepsilon \partial_{xxx}^3 u_{\varepsilon,\delta}}{\sqrt{3}} \right| dx \\
&\leq 3 \int_{\mathbb{R}} (f''(u_{\varepsilon,\delta}))^2 (\partial_x u_{\varepsilon,\delta})^4 dx + \frac{\varepsilon^2}{3} \|\partial_{xxx}^3 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq 3 \|f''\|_{L^\infty(I_{T,2})}^2 \|\partial_x u_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})}^2 \|\partial_x u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon^2}{3} \|\partial_{xxx}^3 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq 3 \|f''\|_{L^\infty(I_{T,2})}^2 C(T) \|\partial_x u_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})}^2 \\
&\quad + \frac{\varepsilon^2}{3} \|\partial_{xxx}^3 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2, \\
2\varepsilon \int_{\mathbb{R}} f'(u_{\varepsilon,\delta}) \partial_{xx}^2 u_{\varepsilon,\delta} \partial_{xxx}^3 u_{\varepsilon,\delta} dx \\
&\leq 2\varepsilon \left| \int_{\mathbb{R}} f'(u_{\varepsilon,\delta}) \partial_{xx}^2 u_{\varepsilon,\delta} \partial_{xxx}^3 u_{\varepsilon,\delta} dx \right| \\
&\leq 2 \int_{\mathbb{R}} \left| \sqrt{3} f'(u_{\varepsilon,\delta}) \partial_{xx}^2 u_{\varepsilon,\delta} \right| \left| \frac{\varepsilon \partial_{xxx}^3 u_{\varepsilon,\delta}}{\sqrt{3}} \right| dx \\
&\leq 3 \int_{\mathbb{R}} (f'(u_{\varepsilon,\delta}))^2 (\partial_{xx}^2 u_{\varepsilon,\delta})^2 dx + \frac{\varepsilon^2}{3} \|\partial_{xxx}^3 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq 3 \|f'\|_{L^\infty(I_{T,2})}^2 \|\partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \frac{\varepsilon^2}{3} \|\partial_{xxx}^3 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

where $I_{T,1}$ is defined in (2.34) and $I_{T,2}$ is defined in (2.62). Therefore,

$$\begin{aligned}
&\frac{d}{dt} \left(\varepsilon \|\partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + 2\varepsilon^2 \|\partial_{xxx}^3 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq \varepsilon^2 \|\partial_{xxx}^3 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\quad + 3 \|f''\|_{L^\infty(I_{T,2})}^2 C(T) \|\partial_x u_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})}^2 \\
&\quad + 3 \|f'\|_{L^\infty(I_{T,2})}^2 \|\partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(T),
\end{aligned}$$

that is

$$\begin{aligned}
&\frac{d}{dt} \left(\varepsilon \|\partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) + \varepsilon^2 \|\partial_{xxx}^3 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\
&\leq C(T) \|\partial_x u_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})}^2 + C(T) \\
&\quad + C(T) \|\partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

An integration on $(0, t)$, (2.7) and (2.60) give

$$\begin{aligned}
&\varepsilon \|\partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \int_0^t \|\partial_{xxx}^3 u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
&\leq \left(C(T) \|\partial_x u_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})}^2 + C(T) \right) \int_0^t ds \\
&\quad + C(T) \int_0^t \|\partial_{xx}^2 u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
&\leq C(T) \|\partial_x u_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})}^2 + C(T).
\end{aligned}$$

Thus,

$$(2.67) \quad \begin{aligned} & \varepsilon \left\| \partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \int_0^t \left\| \partial_{xxx}^3 u_{\varepsilon,\delta}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\ & \leq C(T) \left(1 + \left\| \partial_x u_{\varepsilon,\delta} \right\|_{L^\infty(I_{T,1})}^2 \right). \end{aligned}$$

Due to (2.60), (2.67) and the Hölder inequality,

$$\begin{aligned} (\partial_x u_{\varepsilon,\delta}(t, x))^2 & \leq 2 \int_{\mathbb{R}} |\partial_x u_{\varepsilon,\delta}| |\partial_{xx}^2 u_{\varepsilon,\delta}| dx \\ & \leq 2 \left\| \partial_x u_{\varepsilon,\delta}(t, \cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_{xx}^2 u_{\varepsilon,\delta}(t, \cdot) \right\|_{L^2(\mathbb{R})} \\ & \leq C(T) \sqrt{\left(1 + \left\| \partial_x u_{\varepsilon,\delta} \right\|_{L^\infty(I_{T,1})}^2 \right)}. \end{aligned}$$

Then,

$$\left\| \partial_x u_{\varepsilon,\delta} \right\|_{L^\infty(I_{T,1})}^4 - C(T) \left\| \partial_x u_{\varepsilon,\delta} \right\|_{L^\infty(I_{T,1})}^2 - C(T) \leq 0,$$

which gives (2.64).

(2.65) follows from (2.64) and (2.67). \square

Arguing as in [6], we obtain the following result

Lemma 2.9. *Let $T > 0$, $\ell > 2$ and $0 < \delta < 1$. For each $t \in (0, T)$,*

$$(2.68) \quad \partial_x^\ell u_{\varepsilon,\delta}(t, \cdot) \in L^2(\mathbb{R}).$$

We are in a position to state and prove the following result.

Lemma 2.10. *Let $T > 0$. Assume (1.6), (2.3), (2.5) and (2.4). Then there exist*

$$(2.69) \quad u_\varepsilon \in L^\infty((0, T) \times \mathbb{R}) \cap C((0, T); H^\ell(\mathbb{R})), \quad \ell > 2,$$

$$(2.70) \quad P_\varepsilon \in L^\infty((0, T) \times \mathbb{R}) \cap L^2((0, T) \times \mathbb{R}),$$

where u_ε is a classic solution of the Cauchy problem of (2.1).

Proof. Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be any convex C^2 entropy function, and $q : \mathbb{R} \rightarrow \mathbb{R}$ be the corresponding entropy flux defined by $q' = f'\eta'$. By multiplying the first equation in (2.6) with $\eta'(u_\varepsilon)$ and using the chain rule, we get

$$\partial_t \eta(u_{\varepsilon,\delta}) + \partial_x q(u_{\varepsilon,\delta}) = \underbrace{\varepsilon \partial_{xx}^2 \eta(u_{\varepsilon,\delta})}_{=:\mathcal{L}_{1,\delta}} \underbrace{- \varepsilon \eta''(u_{\varepsilon,\delta}) (\partial_x u_{\varepsilon,\delta})^2}_{=:\mathcal{L}_{2,\delta}} \underbrace{+ \gamma \eta'(u_{\varepsilon,\delta}) P_{\varepsilon,\delta}}_{=:\mathcal{L}_{3,\delta}},$$

where $\mathcal{L}_{1,\delta}$, $\mathcal{L}_{2,\delta}$, $\mathcal{L}_{3,\delta}$ are distributions.

Let us show that

$$(2.71) \quad \{\mathcal{L}_{1,\delta}\}_\delta \text{ is compact in } H^{-1}((0, T) \times \mathbb{R}), \quad T > 0.$$

Since

$$\varepsilon \partial_{xx}^2 \eta(u_{\varepsilon,\delta}) = \partial_x (\varepsilon \eta'(u_{\varepsilon,\delta}) \partial_x u_{\varepsilon,\delta}),$$

we have to prove that

$$(2.72) \quad \{\varepsilon \eta'(u_{\varepsilon,\delta}) \partial_x u_{\varepsilon,\delta}\}_\delta \text{ is bounded in } L^2((0, T) \times \mathbb{R}), \quad T > 0,$$

$$(2.73) \quad \{\varepsilon \eta''(u_{\varepsilon,\delta}) (\partial_x u_{\varepsilon,\delta})^2 + \varepsilon \eta'(u_{\varepsilon,\delta}) \partial_{xx}^2 u_{\varepsilon,\delta}\}_\delta \text{ is bounded in } L^2((0, T) \times \mathbb{R}), \quad T > 0.$$

We begin by proving that (2.72) holds true. Thanks to Lemmas 2.3 and 2.6,

$$\left\| \varepsilon \eta'(u_{\varepsilon,\delta}) \partial_x u_{\varepsilon,\delta} \right\|_{L^2((0,T) \times \mathbb{R})}^2 \leq \varepsilon^2 \left\| \eta' \right\|_{L^\infty(I_{T,2})}^2 \int_0^T \left\| \partial_x u_{\varepsilon,\delta}(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds$$

$$\begin{aligned}
&\leq \varepsilon^2 \|\eta'\|_{L^\infty(I_{T,2})}^2 e^{2\gamma T} \int_0^T e^{-2\gamma s} \|\partial_x u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
&\leq \frac{\varepsilon}{2} \|\eta'\|_{L^\infty(I_{T,2})}^2 e^{2\gamma T} \|u_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2 \leq C(T),
\end{aligned}$$

where $I_{T,2}$ is defined in (2.62).

We claim that

$$(2.74) \quad \{\varepsilon \eta''(u_{\varepsilon,\delta})(\partial_x u_{\varepsilon,\delta})^2\}_\delta \text{ is bounded in } L^2((0, T) \times \mathbb{R}).$$

Due to Lemmas 2.3, 2.6, 2.8

$$\begin{aligned}
\|\varepsilon \eta''(u_{\varepsilon,\delta})(\partial_x u_{\varepsilon,\delta})^2\|_{L^2((0,T) \times \mathbb{R})}^2 &\leq \varepsilon^2 \|\eta''\|_{L^\infty(I_{T,2})}^2 \int_0^T \int_{\mathbb{R}} (\partial_x u_{\varepsilon,\delta}(s, x))^4 ds dx \\
&\leq \varepsilon^2 \|\eta''\|_{L^\infty(I_{T,2})}^2 \|\partial_x u_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})}^2 \int_0^T \|\partial_x u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
&\leq \frac{\varepsilon}{2} \|\eta''\|_{L^\infty(I_{T,2})}^2 \|\partial_x u_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})}^2 e^{2\gamma T} \|u_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2 \leq C(T),
\end{aligned}$$

where $I_{T,1}$ is defined in (2.34).

We claim that

$$(2.75) \quad \{\varepsilon \eta'(u_{\varepsilon,\delta}) \partial_{xx}^2 u_{\varepsilon,\delta}\}_\delta \text{ is bounded in } L^2((0, T) \times \mathbb{R}).$$

Thanks to Lemmas 2.6 and 2.7,

$$\begin{aligned}
\|\varepsilon \eta'(u_{\varepsilon,\delta}) \partial_{xx}^2 u_{\varepsilon,\delta}\|_{L^2((0,T) \times \mathbb{R})}^2 &\leq \varepsilon^2 \|\eta'\|_{L^\infty(I_{T,2})}^2 \int_0^T \|\partial_{xx}^2 u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
&\leq \varepsilon^2 \|\eta'\|_{L^\infty(I_{T,2})}^2 C(T) \leq C(T).
\end{aligned}$$

(2.74) and (2.75) give (2.73).

Therefore, (2.71) follows from (2.72) and (2.73).

We have that

$$\{\mathcal{L}_{2,\delta}\}_{\delta>0} \text{ is bounded in } L^1((0, T) \times \mathbb{R}).$$

Due to Lemmas 2.3, 2.6,

$$\begin{aligned}
\|\varepsilon \eta''(u_{\varepsilon,\delta})(\partial_x u_{\varepsilon,\delta})^2\|_{L^1((0,T) \times \mathbb{R})} &\leq \varepsilon \|\eta''\|_{L^\infty(I_{T,2})} \int_0^T \|\partial_x u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
&\leq \varepsilon \|\eta''\|_{L^\infty(I_{T,2})}^2 e^{2\gamma T} \int_0^T e^{-2\gamma s} \|\partial_x u_{\varepsilon,\delta}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\
&\leq \frac{\|\eta''\|_{L^\infty(I_{T,2})}^2 e^{2\gamma T}}{2} \|u_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2 \leq C(T).
\end{aligned}$$

We have that

$$\{\mathcal{L}_{3,\delta}\}_{\delta>0} \text{ is bounded in } L_{loc}^1((0, T) \times \mathbb{R}).$$

Let K be a compact subset of $(0, T) \times \mathbb{R}$. For Lemmas 2.5 and 2.6,

$$\begin{aligned}
\|\gamma \eta'(u_{\varepsilon,\delta}) P_{\varepsilon,\delta}\|_{L^1(K)} &= \gamma \int_K |\eta'(u_{\varepsilon,\delta})| |P_{\varepsilon,\delta}| dt dx \\
&\leq \gamma \|\eta'\|_{L^\infty(I_{T,2})} \|P_{\varepsilon,\delta}\|_{L^\infty(I_{T,1})} |K|.
\end{aligned}$$

Therefore, Murat's lemma [22] implies that

$$(2.76) \quad \{\partial_t \eta(u_{\varepsilon,\delta}) + \partial_x q(u_{\varepsilon,\delta})\}_{\delta>0} \text{ lies in a compact subset of } H_{loc}^{-1}((0, \infty) \times \mathbb{R}).$$

The L^∞ bound stated in Lemma 2.6, (2.76) and the Tartar's compensated compactness method [30] give the existence of a subsequence $\{u_{\varepsilon, \delta_k}\}_{k \in \mathbb{N}}$ and a limit function $u_\varepsilon \in L^\infty((0, T) \times \mathbb{R})$ such that

$$(2.77) \quad u_{\varepsilon, \delta_k} \rightarrow u_\varepsilon \text{ a.e. and in } L^p_{loc}((0, T) \times \mathbb{R}), \quad 1 \leq p < \infty.$$

Hence,

$$(2.78) \quad u_{\varepsilon, \delta_k} \rightarrow u_\varepsilon \text{ in } L^\infty((0, T) \times \mathbb{R}).$$

Moreover, for convexity, we have

$$(2.79) \quad \begin{aligned} & \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon e^{2\gamma t} \int_0^t e^{-2\gamma s} \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \\ & \varepsilon \|\partial_x u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \int_0^t \|\partial_{xx}^2 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \\ & \varepsilon \|\partial_{xx}^2 u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \int_0^t \|\partial_{xxx}^3 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T). \end{aligned}$$

We need only to observe that

$$\begin{aligned} & 2\varepsilon e^{2\gamma t} \int_0^t e^{-2\gamma s} \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ & \leq 2\varepsilon e^{2\gamma t} \liminf_k \int_0^t e^{-2\gamma s} \|\partial_x u_{\varepsilon, \delta_k}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \\ & \varepsilon^2 \int_0^t \|\partial_{xx}^2 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq \varepsilon^2 \liminf_k \int_0^t \|\partial_{xx}^2 u_{\varepsilon, \delta_k}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T), \\ & \varepsilon^2 \int_0^t \|\partial_{xxx}^3 u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq \varepsilon^2 \liminf_k \int_0^t \|\partial_{xxx}^3 u_{\varepsilon, \delta_k}(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq C(T). \end{aligned}$$

Moreover, it follows from convexity and Lemma 2.9 that

$$(2.80) \quad \partial_x^\ell u_\varepsilon(t, \cdot) \in L^2(\mathbb{R}), \quad \ell > 2, \quad t \in (0, T).$$

Therefore, (2.78), (2.79) and (2.80) give (2.69). (2.70) follows from Lemma (2.5).

Finally, we prove that

$$(2.81) \quad \int_{-\infty}^x u_\varepsilon(t, y) dy = P_\varepsilon(t, x), \quad \text{a.e. in } (t, x) \in I_{T,1}.$$

Integrating the second equation of (2.6) on $(-\infty, x)$, for (2.10), we have that

$$(2.82) \quad \int_{-\infty}^x u_{\varepsilon, \delta_k}(t, y) dy = P_{\varepsilon, \delta_k}(t, x) - \delta_k \partial_x P_{\varepsilon, \delta_k}(t, x).$$

We show that

$$(2.83) \quad \delta \partial_x P_{\varepsilon, \delta}(t, x) \rightarrow 0 \text{ in } L^\infty(0, T; L^\infty(0, \infty)), \quad T > 0 \text{ as } \delta \rightarrow 0.$$

It follows from (2.18) that

$$\delta \|\partial_x P_{\varepsilon, \delta}\|_{L^\infty(0, T; L^\infty(0, \infty))} \leq \sqrt{\delta} e^{\gamma t} \|u_{\varepsilon, 0}\|_{L^2(\mathbb{R})} = \sqrt{\delta} C(T) \rightarrow 0,$$

that is (2.83).

Therefore, (2.81) follows from (2.69), (2.70), (2.82) and (2.83). The proof is done. \square

Lemma 2.11. *Let $u_\varepsilon(t, x)$ be a classic solution of (2.1). Then,*

$$(2.84) \quad \int_{\mathbb{R}} u_\varepsilon(t, x) dx = 0, \quad t \geq 0,$$

Proof. Differentiating (2.1) with respect to x , we have

$$(2.85) \quad \partial_x(\partial_t u_\varepsilon + \partial_x f(u_\varepsilon) - \varepsilon \partial_{xx}^2 u_\varepsilon) = \gamma u_\varepsilon.$$

Since u_ε is a smooth solution of (2.1), an integration over \mathbb{R} gives (2.84). \square

We are ready for the proof of Theorem 2.1.

Proof of Theorem 2.1. Lemma 2.10 gives the existence of a classic solution $u_\varepsilon(t, x)$ of (2.1), or (2.2).

Let us show that $u_\varepsilon(t, x)$ is unique and (2.9) holds true. Let $u_\varepsilon, v_\varepsilon$ be two classic solution of (2.1), or (2.2), that is

$$\begin{cases} \partial_t u_\varepsilon + f'(u_\varepsilon) \partial_x u_\varepsilon = \gamma P_\varepsilon^{u_\varepsilon} + \varepsilon \partial_{xx}^2 u_\varepsilon, & t > 0, x \in \mathbb{R}, \\ \partial_x P_\varepsilon^{u_\varepsilon} = u_\varepsilon, & t > 0, x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases}$$

$$\begin{cases} \partial_t v_\varepsilon + f'(v_\varepsilon) \partial_x v_\varepsilon = \gamma P_\varepsilon^{v_\varepsilon} + \varepsilon \partial_{xx}^2 v_\varepsilon, & t > 0, x \in \mathbb{R}, \\ \partial_x P_\varepsilon^{v_\varepsilon} = v_\varepsilon, & t > 0, x \in \mathbb{R}, \\ v_\varepsilon(0, x) = v_{\varepsilon,0}(x), & x \in \mathbb{R}. \end{cases}$$

Then, the function

$$(2.86) \quad \omega_\varepsilon(t, x) = u_\varepsilon(t, x) - v_\varepsilon(t, x)$$

is solution of the following Cauchy problem

$$(2.87) \quad \begin{cases} \partial_t \omega_\varepsilon + f'(u_\varepsilon) \partial_x u_\varepsilon - f'(v_\varepsilon) \partial_x v_\varepsilon = \gamma \Omega_\varepsilon + \varepsilon \partial_{xx}^2 \omega_\varepsilon, & t > 0, x \in \mathbb{R}, \\ \partial_x \Omega_\varepsilon = \omega_\varepsilon, & t > 0, x \in \mathbb{R}, \\ \omega_\varepsilon(0, x) = u_{\varepsilon,0}(x) - v_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases}$$

where

$$(2.88) \quad \begin{aligned} \Omega_\varepsilon(t, x) &= P_\varepsilon^{u_\varepsilon}(t, x) - P_\varepsilon^{v_\varepsilon}(t, x) \\ &= \int_{-\infty}^x u_\varepsilon(t, y) dy - \int_{-\infty}^x v_\varepsilon(t, y) dy \\ &= \int_{-\infty}^x (u_\varepsilon(t, y) - v_\varepsilon(t, y)) dy = \int_{-\infty}^x \omega_\varepsilon(t, y) dy. \end{aligned}$$

It follows from Lemma 2.11 and (2.88) that

$$(2.89) \quad \Omega_\varepsilon(t, \infty) = \int_{\mathbb{R}} u_\varepsilon(t, y) dy - \int_{\mathbb{R}} v_\varepsilon(t, y) dy = 0.$$

Observe that, for (2.86),

$$\begin{aligned} f'(u_\varepsilon) \partial_x u_\varepsilon - f'(v_\varepsilon) \partial_x v_\varepsilon &= f'(u_\varepsilon) \partial_x u_\varepsilon - f'(u_\varepsilon) \partial_x v_\varepsilon + f'(u_\varepsilon) \partial_x v_\varepsilon - f'(v_\varepsilon) \partial_x v_\varepsilon \\ &= f'(u_\varepsilon) \partial_x (u_\varepsilon - v_\varepsilon) + \partial_x v_\varepsilon (f'(u_\varepsilon) - f'(v_\varepsilon)) \\ &= f'(u_\varepsilon) \partial_x \omega_\varepsilon + (f'(u_\varepsilon) - f'(v_\varepsilon)) \partial_x v_\varepsilon. \end{aligned}$$

Therefore, the first equation of (2.87) is equivalent to the following one:

$$(2.90) \quad \partial_t \omega_\varepsilon + f'(u_\varepsilon) \partial_x \omega_\varepsilon + (f'(u_\varepsilon) - f'(v_\varepsilon)) \partial_x v_\varepsilon = \gamma \Omega_\varepsilon + \varepsilon \partial_{xx}^2 \omega_\varepsilon.$$

Moreover, since u_ε and v_ε are in $L^\infty((0, T) \times \mathbb{R})$, we have that

$$(2.91) \quad \left| f'(u_\varepsilon(t, x)) - f'(v_\varepsilon(t, x)) \right| \leq C(T) |u_\varepsilon(t, x) - v_\varepsilon(t, x)|, \quad (t, x) \in (0, T) \times \mathbb{R},$$

where

$$(2.92) \quad C(T) = \sup_{(0,T) \times \mathbb{R}} \left\{ |f''(u_\varepsilon)| + |f''(v_\varepsilon)| \right\}.$$

Therefore, (2.86) and (2.91) give

$$(2.93) \quad \left| f'(u_\varepsilon(t, x)) - f'(v_\varepsilon(t, x)) \right| \leq C(T) |\omega_\varepsilon(t, x)|, \quad (t, x) \in (0, T) \times \mathbb{R}$$

Multiplying (2.90) by ω_ε , and integration on \mathbb{R} gives

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \omega_\varepsilon^2 dx &= 2 \int_{\mathbb{R}} \omega_\varepsilon \partial_t \omega_\varepsilon dx \\ &= 2\varepsilon \int_{\mathbb{R}} \omega_\varepsilon \partial_{xx}^2 \omega_\varepsilon dx - 2 \int_{\mathbb{R}} \omega_\varepsilon f'(u_\varepsilon) \partial_x \omega_\varepsilon dx \\ &\quad - 2 \int_{\mathbb{R}} \omega_\varepsilon (f'(u_\varepsilon) - f'(v_\varepsilon)) \partial_x v_\varepsilon dx + 2\gamma \int_{\mathbb{R}} \Omega_\varepsilon \omega_\varepsilon dx \\ &= -2\varepsilon \int_{\mathbb{R}} (\partial_x \omega_\varepsilon)^2 dx + \int_{\mathbb{R}} \omega_\varepsilon^2 f''(u_\varepsilon) \partial_x u_\varepsilon dx \\ &\quad - 2 \int_{\mathbb{R}} \omega_\varepsilon (f'(u_\varepsilon) - f'(v_\varepsilon)) \partial_x v_\varepsilon dx + 2\gamma \int_{\mathbb{R}} \Omega_\varepsilon \omega_\varepsilon dx. \end{aligned}$$

It follows from the second equation of (2.87) and Lemma 2.11 that

$$(2.94) \quad \begin{aligned} &\frac{d}{dt} \|\omega_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x \omega_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\leq \int_{\mathbb{R}} \omega_\varepsilon^2 |f''(u_\varepsilon)| |\partial_x u_\varepsilon| dx + 2 \int_{\mathbb{R}} |\omega_\varepsilon| |(f'(u_\varepsilon) - f'(v_\varepsilon))| |\partial_x v_\varepsilon| dx. \end{aligned}$$

Since $u_\varepsilon(t, \cdot), v_\varepsilon(t, \cdot) \in H^\ell(\mathbb{R}), \ell > 2$, for each $t \in (0, T)$, then

$$(2.95) \quad \partial_x u_\varepsilon(t, \cdot), \partial_x v_\varepsilon(t, \cdot) \in H^{\ell-1}(\mathbb{R}) \subset L^\infty(\mathbb{R}), \quad t \in (0, T).$$

Therefore, thanks to (2.91), (2.92), (2.94) and (2.95),

$$\frac{d}{dt} \|\omega_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|\partial_x \omega_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq C(T) \|\omega_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2.$$

The Gronwall Lemma gives

$$(2.96) \quad \|\omega_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon e^{C(T)t} \int_0^t e^{-C(T)s} \|\partial_x \omega_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq e^{C(T)t} \|\omega_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2.$$

Hence, (2.9) follows from (2.86), (2.87) and (2.96). \square

3. EXISTENCE OF ENTROPY SOLUTIONS FOR OSTROVSHY-HUNTER EQUATION

This section is devoted to the existence of entropy solutions for (1.8), or (1.9).

Fix a small number $\varepsilon > 0$, and let $u_\varepsilon = u_\varepsilon(t, x)$ be the unique classical solution of (2.1), where $u_{\varepsilon,0}$ is a $C^\infty(\mathbb{R})$ approximation of u_0 such that

$$(3.1) \quad \begin{aligned} &\|u_{\varepsilon,0}\|_{L^2(\mathbb{R})} \leq \|u_0\|_{L^2(\mathbb{R})}, \quad \|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}, \\ &\int_{\mathbb{R}} u_{\varepsilon,0}(x) dx = 0, \quad \int_{\mathbb{R}} \left(\int_{-\infty}^x u_{\varepsilon,0}(y) dy \right)^2 dx \leq \|P_0\|_{L^2(\mathbb{R})}^2, \\ &\int_{\mathbb{R}} \left(\int_{-\infty}^x u_{\varepsilon,0}(y) dy \right) dx = \int_{\mathbb{R}} P_{\varepsilon,0}(x) dx = 0, \end{aligned}$$

where

$$(3.2) \quad P_{\varepsilon,0}(x) = \int_{-\infty}^x u_{\varepsilon,0}(y)dy,$$

and $\|P_0\|_{L^2(\mathbb{R})}$ is defined in (1.5).

Let us prove some a priori estimates on u_ε and P_ε , denoting with $C(T)$ the constants which depend on T , but independent on ε .

Following [5, Lemma 6], or [9, Lemma 2.3.1], we show this result.

Lemma 3.1. *Let us suppose that, for each $t \geq 0$,*

$$(3.3) \quad P_\varepsilon(t, x) \text{ is integrable at } -\infty, \text{ (or at } +\infty),$$

where $P_\varepsilon(t, x)$ is defined in (2.1). Then, the following statements are equivalent:

$$(3.4) \quad \int_{\mathbb{R}} u_\varepsilon(t, x)dx = 0,$$

$$(3.5) \quad \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds = \|u_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2,$$

$$(3.6) \quad \int_{\mathbb{R}} P_\varepsilon(t, x)dx = 0,$$

$$(3.7) \quad \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 = \|P_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2 - 2 \int_0^t \int_{\mathbb{R}} P_\varepsilon f(u_\varepsilon) ds dx,$$

for every $t \geq 0$.

Proof. Let $t > 0$. We begin by proving that (3.4) implies (3.5).

Multiplying (2.2) by u_ε , an integration on \mathbb{R} gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} u_\varepsilon^2 dx &= \int_{\mathbb{R}} u_\varepsilon \partial_t u_\varepsilon dx \\ &= \varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_{xx}^2 u_\varepsilon dx - \int_{\mathbb{R}} u_\varepsilon f'(u_\varepsilon) \partial_x u_\varepsilon dx + \gamma \int_{\mathbb{R}} u_\varepsilon \left(\int_{-\infty}^x u_\varepsilon dy \right) dx \\ &= -\varepsilon \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 dx + \gamma \int_{\mathbb{R}} u_\varepsilon \left(\int_{-\infty}^x u_\varepsilon dy \right) dx. \end{aligned}$$

For (2.1),

$$\int_{\mathbb{R}} u_\varepsilon \left(\int_{-\infty}^x u_\varepsilon dy \right) dx = \int_{\mathbb{R}} P_\varepsilon(t, x) \partial_x P_\varepsilon(t, x) dx = \frac{1}{2} P_\varepsilon^2(t, \infty).$$

Then,

$$(3.8) \quad \frac{d}{dt} \int_{\mathbb{R}} u_\varepsilon^2 dx + 2\varepsilon \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 dx = \gamma P_\varepsilon^2(t, \infty).$$

Thanks to (3.4),

$$(3.9) \quad \lim_{x \rightarrow \infty} P_\varepsilon^2(t, x) = \left(\int_{\mathbb{R}} u_\varepsilon(t, x) dx \right)^2 = 0.$$

(3.8), (3.9) and an integration on $(0, t)$ give (3.5).

Let us show that (3.5) implies (3.4). We assume by contradiction that (3.4) does not hold, namely:

$$(3.10) \quad \int_{\mathbb{R}} u_\varepsilon(t, x) dx \neq 0.$$

For (2.1),

$$P_\varepsilon^2(t, \infty) = \left(\int_{\mathbb{R}} u_\varepsilon(t, x) dx \right)^2 \neq 0.$$

Therefore, (3.8) and an integration on $(0, t)$ give

$$\|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \neq \|u_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2,$$

which is in contradiction with (3.5).

Let us show that (3.4) implies (3.6). We begin by observing that, for (3.3), we can consider the following function:

$$(3.11) \quad F_\varepsilon(t, x) = \int_{-\infty}^x P_\varepsilon(t, y) dy.$$

Thanks to the regularity of u_ε and (3.11), integrating on $(-\infty, x)$ the first equation of (2.1), we get

$$\int_{-\infty}^x \partial_t u_\varepsilon(t, y) dy + f(u_\varepsilon(t, x)) - f(0) - \varepsilon \partial_x u_\varepsilon(t, x) = \gamma F_\varepsilon(t, x),$$

that is

$$(3.12) \quad \frac{d}{dt} \int_{-\infty}^x u_\varepsilon(t, y) dy + f(u_\varepsilon(t, x)) - f(0) - \varepsilon \partial_x u_\varepsilon(t, x) = \gamma F_\varepsilon(t, x).$$

Instead, from the second equation of (2.1), we have

$$(3.13) \quad \partial_t P_\varepsilon(t, x) = \frac{d}{dt} \int_{-\infty}^x u_\varepsilon(t, y) dy.$$

It follows from (3.12) and (3.13) that

$$(3.14) \quad \partial_t P_\varepsilon(t, x) + f(u_\varepsilon(t, x)) - f(0) + \varepsilon \partial_x u_\varepsilon(t, x) = \gamma F_\varepsilon(t, x).$$

We observe that, for (3.4) and (3.13),

$$(3.15) \quad \lim_{x \rightarrow \infty} \partial_t P_\varepsilon(t, x) = \int_{\mathbb{R}} \partial_t u_\varepsilon(t, x) dx = \frac{d}{dt} \int_{\mathbb{R}} u_\varepsilon(t, x) dx = 0,$$

while for the regularity of u_ε ,

$$(3.16) \quad \lim_{x \rightarrow \infty} \left(f(u_\varepsilon(t, x)) - f(0) - \varepsilon \partial_x u_\varepsilon(t, x) \right) = 0.$$

Therefore, for (3.11), (3.14), (3.15) and (3.16), we get

$$(3.17) \quad F_\varepsilon(t, \infty) = \int_{\mathbb{R}} P_\varepsilon(t, x) dx = 0,$$

that is (3.6).

Let us show that (3.6) implies (3.4). We assume by contradiction that (3.4) does not hold, that is (3.10). Then, for (3.13),

$$(3.18) \quad \lim_{x \rightarrow \infty} \partial_t P_\varepsilon(t, x) = \int_{\mathbb{R}} \partial_t u_\varepsilon(t, x) dx = \frac{d}{dt} \int_{\mathbb{R}} u_\varepsilon(t, x) dx \neq 0.$$

It follows from (3.11), (3.14), (3.16) and (3.18) that

$$\int_{\mathbb{R}} P_\varepsilon(t, x) dx \neq 0,$$

which is in contradiction with (3.6).

Let us show that (3.6) implies (3.7). Multiplying (3.14) by P_ε , an integration on \mathbb{R} gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} P_\varepsilon^2 dx &= \int_{\mathbb{R}} P_\varepsilon \partial_t P_\varepsilon dx \\ &= \varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon P_\varepsilon dx - \int_{\mathbb{R}} P_\varepsilon f(u_\varepsilon) dx + f(0) \int_{\mathbb{R}} P_\varepsilon dx + \gamma \int_{\mathbb{R}} P_\varepsilon F_\varepsilon dx, \end{aligned}$$

that is

$$\frac{d}{dt} \int_{\mathbb{R}} P_\varepsilon^2 dx = 2\varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon P_\varepsilon dx - 2 \int_{\mathbb{R}} P_\varepsilon f(u_\varepsilon) dx + f(0) \int_{\mathbb{R}} P_\varepsilon dx + 2\gamma \int_{\mathbb{R}} P_\varepsilon F_\varepsilon dx.$$

For (2.1),

$$2\varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon P_\varepsilon dx = -2\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x P_\varepsilon dx = -2\varepsilon \int_{\mathbb{R}} u_\varepsilon^2 dx,$$

while for (3.11),

$$2 \int_{\mathbb{R}} P_\varepsilon F_\varepsilon dx = 2 \int_{\mathbb{R}} F_\varepsilon \partial_x F_\varepsilon dx = F_\varepsilon^2(t, \infty) - F_\varepsilon^2(t, -\infty) = F_\varepsilon^2(t, \infty).$$

Then,

$$(3.19) \quad \frac{d}{dt} \int_{\mathbb{R}} P_\varepsilon^2 dx = -2\varepsilon \int_{\mathbb{R}} u_\varepsilon^2 dx - 2 \int_{\mathbb{R}} P_\varepsilon f(u_\varepsilon) dx + f(0) \int_{\mathbb{R}} P_\varepsilon dx + \gamma F_\varepsilon^2(t, \infty).$$

Thanks to (3.6),

$$(3.20) \quad \begin{aligned} \lim_{x \rightarrow \infty} F_\varepsilon^2(t, x) &= \left(\int_{\mathbb{R}} P_\varepsilon(t, x) dx \right)^2 = 0, \\ f(0) \int_{\mathbb{R}} P_\varepsilon dx &= 0. \end{aligned}$$

(3.19), (3.20) and an integration on $(0, t)$ give (3.7).

Let us show that (3.7) implies (3.6). We assume by contradiction that (3.6) does not hold, namely:

$$\int_{\mathbb{R}} P_\varepsilon(t, x) dx \neq 0.$$

For (3.11),

$$(3.21) \quad F_\varepsilon^2(t, \infty) = \left(\int_{\mathbb{R}} P_\varepsilon(t, x) dx \right)^2 \neq 0.$$

Moreover,

$$(3.22) \quad f(0) \int_{\mathbb{R}} P_\varepsilon dx \neq 0.$$

Therefore, (3.19), (3.21) and (3.22) gives

$$\frac{d}{dt} \int_{\mathbb{R}} P_\varepsilon^2 dx + 2\varepsilon \int_{\mathbb{R}} u_\varepsilon^2 dx + \int_{\mathbb{R}} P_\varepsilon u_\varepsilon^2 dx \neq 0,$$

that is

$$\frac{d}{dt} \int_{\mathbb{R}} P_\varepsilon^2 dx \neq -2\varepsilon \int_{\mathbb{R}} u_\varepsilon^2 dx - \int_{\mathbb{R}} P_\varepsilon f(u_\varepsilon) dx.$$

Therefore, we have that

$$\|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \neq \|P_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2 - 2 \int_0^t \int_{\mathbb{R}} P_\varepsilon f(u_\varepsilon) ds dx,$$

which is a contradiction with (3.7). \square

Lemma 3.2. *For each $t \geq 0$, (3.3) and (3.6) hold true. Moreover, we have that*

$$(3.23) \quad \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \leq \|u_0\|_{L^2(\mathbb{R})}^2,$$

$$(3.24) \quad \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \|P_0\|_{L^2(\mathbb{R})}^2 + 2C_0 \int_0^t \int_{\mathbb{R}} |P_\varepsilon| u_\varepsilon^2 ds dx.$$

Proof. We begin by proving that (3.3) holds true. Let a be, an arbitrary real number. Integrating on (a, x) the second equation of (2.1), we get

$$\int_a^x u_\varepsilon(t, y) dy = P_\varepsilon(t, x) - P_\varepsilon(t, a).$$

Since $P_\varepsilon(t, -\infty) = 0$, then

$$(3.25) \quad \int_a^{-\infty} u_\varepsilon(t, x) dx = -P_\varepsilon(t, a).$$

Differentiating (3.25) with respect to t , we get

$$(3.26) \quad \frac{d}{dt} \int_a^{-\infty} u_\varepsilon(t, x) dx = \int_a^{-\infty} \partial_t u_\varepsilon(t, x) dx = -\partial_t P_\varepsilon(t, a).$$

Integrating on (a, x) the first equation of (2.1), we obtain that

$$(3.27) \quad \begin{aligned} \int_a^x \partial_t u_\varepsilon(t, y) dy + f(u_\varepsilon(t, x)) - f(u_\varepsilon(t, a)) \\ - \varepsilon \partial_x u_\varepsilon(t, x) + \varepsilon \partial_x u_\varepsilon(t, a) = \gamma \int_a^x P_\varepsilon(t, y) dy. \end{aligned}$$

Being u_ε a smooth solution of (2.2), we have that

$$(3.28) \quad \lim_{x \rightarrow -\infty} \left(f(u_\varepsilon(t, x)) - \varepsilon \partial_x u_\varepsilon(t, x) \right) = f(0).$$

It follows from (3.26), (3.27) and (3.28) that

$$\gamma \int_a^{-\infty} P_\varepsilon(t, x) dx = -\partial_t P_\varepsilon(t, a) + f(0) - f(u_\varepsilon(t, a)) + \varepsilon \partial_x u_\varepsilon(t, a),$$

which gives (3.3). Therefore, for Lemmas 2.11 and 3.1, we have (3.6). Lemmas 2.11 and 3.1 also say that (3.5) holds true. Thus, (3.23) follows from (3.1) and (3.5).

Finally, we prove (3.24). Again by Lemmas 2.11 and 3.1, we get (3.7). Then, for (1.6) and (3.1),

$$\begin{aligned} \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\varepsilon \|u_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 &= \|P_{\varepsilon,0}\|_{L^2(\mathbb{R})}^2 - 2 \int_0^t \int_{\mathbb{R}} P_\varepsilon f(u_\varepsilon) ds dx \\ &\leq \|P_0\|_{L^2(\mathbb{R})}^2 + 2 \left| \int_0^t \int_{\mathbb{R}} P_\varepsilon f(u_\varepsilon) ds dx \right| \\ &\leq \|P_0\|_{L^2(\mathbb{R})}^2 + 2 \int_0^t \int_{\mathbb{R}} |P_\varepsilon| |f(u_\varepsilon)| ds dx \\ &\leq \|P_0\|_{L^2(\mathbb{R})}^2 + 2C_0 \int_0^t \int_{\mathbb{R}} |P_\varepsilon| u_\varepsilon^2 ds dx, \end{aligned}$$

that is (3.24). □

Lemma 3.3. *Let $T > 0$. There exists a function $C(T) > 0$, independent on ε , such that*

$$(3.29) \quad \|P_\varepsilon\|_{L^\infty(I_{T,1})} \leq C(T),$$

$$(3.30) \quad \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(T),$$

where $I_{T,1}$ is defined in (2.34).

Proof. We begin by observing that, for (3.5) and (3.7),

$$\begin{aligned} \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 &\leq \|P_0\|_{L^2(\mathbb{R})}^2 + 2C_0 \|u_0\|_{L^2(\mathbb{R})}^2 \|P_\varepsilon\|_{L^\infty(I_{T,1})} t \\ &\leq \|P_0\|_{L^2(\mathbb{R})}^2 + 2C_0 \|u_0\|_{L^2(\mathbb{R})}^2 T \|P_\varepsilon\|_{L^\infty(I_{T,1})}. \end{aligned}$$

Hence,

$$(3.31) \quad \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \|P_0\|_{L^2(\mathbb{R})}^2 + C_1(T) \|P_\varepsilon\|_{L^\infty(I_{T,1})}.$$

Due to the Hölder inequality, we get

$$P_\varepsilon^2(t, x) \leq 2 \int_{\mathbb{R}} |P_\varepsilon \partial_x P_\varepsilon| dx \leq 2 \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})},$$

that is

$$P_\varepsilon^4(t, x) \leq 4 \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \|\partial_x P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2.$$

For (2.1), (3.5) and (3.31),

$$\begin{aligned} P_\varepsilon^4(t, x) &\leq 4 \|P_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \|u_0\|_{L^2(\mathbb{R})}^2 \\ &\leq 4 \|P_0\|_{L^2(\mathbb{R})}^2 \|u_0\|_{L^2(\mathbb{R})}^2 + 4 \|u_0\|_{L^2(\mathbb{R})}^2 C_1(T) \|P_\varepsilon\|_{L^\infty(I_{T,1})}. \end{aligned}$$

Therefore,

$$(3.32) \quad \|P_\varepsilon\|_{L^\infty(I_{T,1})}^4 - C_2(T) \|P_\varepsilon\|_{L^\infty(I_{T,1})} - 4 \|P_0\|_{L^2(\mathbb{R})}^2 \|u_0\|_{L^2(\mathbb{R})}^2 \leq 0.$$

Let us consider the following function

$$(3.33) \quad g(X) = X^4 - C_2(T)X - A,$$

where

$$(3.34) \quad A = 4 \|P_0\|_{L^2(\mathbb{R})}^2 \|u_0\|_{L^2(\mathbb{R})}^2 > 0.$$

We observe that

$$(3.35) \quad \lim_{X \rightarrow -\infty} g(X) = \infty, \quad g(0) = -A < 0.$$

Since $g'(X) = 4X^3 - C_2(T)$, we have that

$$g \text{ is increasing in } (E(T), \infty),$$

where $E(T) = \left(\frac{C_2(T)}{4}\right)^{\frac{1}{3}} > 0$.

Thus,

$$(3.36) \quad g(E(T)) < g(0) < 0.$$

Moreover,

$$(3.37) \quad \lim_{X \rightarrow \infty} g(X) = \infty.$$

Then, it follows from (3.35), (3.36) and (3.37) that the function g has only two zeros $D(T) < 0 < C(T)$. Therefore, the inequality

$$X^4 - C_2(T)X - A \leq 0$$

is verified when

$$(3.38) \quad D(T) \leq X \leq C(T).$$

Taking $X = \|P_\varepsilon\|_{L^\infty(I_{T,1})}$, we have (3.29).

Finally, (3.30) follows from (3.29) and (3.31). \square

Arguing as Section 2, Lemma 2.6, we obtain the following result

Lemma 3.4. *Let $T > 0$. Then,*

$$(3.39) \quad \|u_\varepsilon\|_{L^\infty(I_{T,1})} \leq \|u_0\|_{L^\infty(\mathbb{R})} + C(T),$$

where $I_{T,1}$ is defined in (2.34).

Let us continue by proving the existence of a distributional solution to (1.1), (1.2) satisfying (1.11).

Lemma 3.5. *Let $T > 0$. There exists a function $u \in L^\infty((0, T) \times \mathbb{R})$ that is a distributional solution of (1.9) and satisfies (1.11) for every convex entropy $\eta \in C^2(\mathbb{R})$.*

We construct a solution by passing to the limit in a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of viscosity approximations (2.1). We use the compensated compactness method [30].

Lemma 3.6. *Let $T > 0$. There exists a subsequence $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and a limit function $u \in L^\infty((0, T) \times \mathbb{R})$ such that*

$$(3.40) \quad u_{\varepsilon_k} \rightarrow u \text{ a.e. and in } L^p_{loc}((0, T) \times \mathbb{R}), \quad 1 \leq p < \infty.$$

Moreover, we have

$$(3.41) \quad P_{\varepsilon_k} \rightarrow P \text{ in } L^\infty((0, T) \times \mathbb{R}) \cap L^2((0, T) \times \mathbb{R})$$

such that

$$(3.42) \quad \partial_x P = u \text{ in the sense of distributions on } [0, \infty) \times \mathbb{R}.$$

Proof. Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be any convex C^2 entropy function, and $q : \mathbb{R} \rightarrow \mathbb{R}$ be the corresponding entropy flux defined by $q' = f'\eta'$. By multiplying the first equation in (2.1) with $\eta'(u_\varepsilon)$ and using the chain rule, we get

$$\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon) = \underbrace{\varepsilon \partial_{xx}^2 \eta(u_\varepsilon)}_{=:\mathcal{L}_{1,\varepsilon}} \underbrace{- \varepsilon \eta''(u_\varepsilon) (\partial_x u_\varepsilon)^2}_{=:\mathcal{L}_{2,\varepsilon}} + \underbrace{\gamma \eta'(u_\varepsilon) P_\varepsilon}_{=:\mathcal{L}_{3,\varepsilon}},$$

where $\mathcal{L}_{1,\varepsilon}$, $\mathcal{L}_{2,\varepsilon}$, $\mathcal{L}_{3,\varepsilon}$ are distributions.

Let us show that

$$\mathcal{L}_{1,\varepsilon} \rightarrow 0 \text{ in } H^{-1}((0, T) \times \mathbb{R}), \quad T > 0.$$

Since

$$\varepsilon \partial_{xx}^2 \eta(u_\varepsilon) = \partial_x (\varepsilon \eta'(u_\varepsilon) \partial_x u_\varepsilon),$$

for Lemmas 3.2 and 3.4,

$$\begin{aligned} \|\varepsilon \eta'(u_\varepsilon) \partial_x u_\varepsilon\|_{L^2((0,T) \times \mathbb{R})}^2 &\leq \varepsilon^2 \|\eta'\|_{L^\infty(J_T)}^2 \int_0^T \|\partial_x u_\varepsilon(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds \\ &\leq \frac{\varepsilon}{2} \|\eta'\|_{L^\infty(J_T)}^2 \|u_0\|_{L^2(\mathbb{R})}^2 \rightarrow 0, \end{aligned}$$

where

$$J_T = \left(-\|u_0\|_{L^\infty(\mathbb{R})} - C(T), \|u_0\|_{L^\infty(\mathbb{R})} + C(T) \right).$$

Arguing as Lemma (2.10), we obtain that

$$\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1((0, T) \times \mathbb{R}), \quad T > 0,$$

$\{\mathcal{L}_{3,\varepsilon}\}_{\varepsilon>0}$ is uniformly bounded in $L^1_{loc}((0,T) \times \mathbb{R})$, $T > 0$.

Therefore, Murat's lemma [22] implies that

$$(3.43) \quad \{\partial_t \eta(u_\varepsilon) + \partial_x q(u_\varepsilon)\}_{\varepsilon>0} \text{ lies in a compact subset of } H^{-1}_{loc}((0,\infty) \times \mathbb{R}).$$

The L^∞ bound stated in Lemma 3.4, (3.43) and the Tartar's compensated compactness method [30] give the existence of a subsequence $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}$ and a limit function $u \in L^\infty((0,T) \times \mathbb{R})$ such that (3.40) holds.

(3.41) follows from Lemma 3.3.

We conclude by proving that (3.42) holds true.

Let $\phi \in C^\infty(\mathbb{R}^2)$ be a test function with compact support. Multiplying by ϕ the second equation of (2.1), we have that

$$\int_0^\infty \int_{\mathbb{R}} \partial_x P_{\varepsilon_k} \phi dt dx = \int_0^\infty \int_{\mathbb{R}} u_{\varepsilon_k} ds dx,$$

that is

$$(3.44) \quad - \int_0^\infty \int_{\mathbb{R}} P_{\varepsilon_k} \partial_x \phi dt dx = \int_0^\infty \int_{\mathbb{R}} u_{\varepsilon_k} \phi ds dx.$$

(3.42) follows from (3.40), (3.41) and (3.44). \square

4. OLEINIK ESTIMATE AND UNIQUENESS OF THE ENTROPY SOLUTION FOR OSTROVSKY-HUNTER EQUATION

In [5, 9], it is proved that the initial value problem (1.8), or (1.9), admits a unique entropy solution, when the flux is assumed Lipschitz continuous. Denoting with $C(T)$ the constants which depends on T , in this section, we prove the following theorem.

Theorem 4.1 (Oleinik estimate). *Fixed $T > 0$. Let us suppose that the flux f is strictly convex, that is*

$$(4.1) \quad f'' \geq c > 0, \quad \text{for some constant } c.$$

Then, there exists a positive constant $C(T)$ such that

$$(4.2) \quad \frac{u(t,x) - u(t,y)}{x - y} \leq C(T) \left(\frac{1}{t} + 1 \right),$$

for almost every $t \in (0,T)$ and $x, y \in \mathbb{R}$, $x \neq y$, where u is the unique entropy weak solution of (1.8), or (1.9).

Proof. Fixed $T > 0$, let u_ε be the solution of (2.1), or (2.2). We claim that there exists a positive constant $C(T)$ such that

$$(4.3) \quad \partial_x u_\varepsilon(t,x) \leq C(T) \left(\frac{1}{t} + 1 \right), \quad t \in (0,T), x \in \mathbb{R}.$$

Differentiating with respect to x the equation in (2.2), we obtain

$$\partial_{tx}^2 u_\varepsilon + f'(u_\varepsilon) \partial_{xx}^2 u_\varepsilon + f''(u_\varepsilon) (\partial_x u_\varepsilon)^2 - \varepsilon \partial_{xxx}^3 u_\varepsilon = \gamma u_\varepsilon.$$

Let us consider the Cauchy problem

$$(4.4) \quad \begin{cases} \partial_t v + f'(u_\varepsilon) \partial_x v + f''(u_\varepsilon) v^2 - \varepsilon \partial_{xx}^2 v = \gamma u_\varepsilon, & t \in (0,T), x \in \mathbb{R}, \\ v(0,x) = \partial_x u_{0,\varepsilon}(x), & x \in \mathbb{R}. \end{cases}$$

Clearly, the solution of (4.4) is $\partial_x u_\varepsilon$. Due to (3.39) and (4.1),

$$\partial_t v + f'(u_\varepsilon) \partial_x v - \varepsilon \partial_{xx}^2 v = \gamma u_\varepsilon - f''(u_\varepsilon) v^2 \leq C(T) - cv^2.$$

Therefore, a supersolution of (4.4) satisfies the following ordinary differential equation

$$(4.5) \quad \frac{dz}{dt} + cz^2 - C(T) = 0, \quad z(0) = \|\partial_x u_{0,\varepsilon}\|_{L^\infty(\mathbb{R})}.$$

We consider the map

$$Z(t) = \frac{1}{ct} + \sqrt{\frac{C(T)}{c}}, \quad t \in (0, T).$$

Observe that

$$\frac{dZ}{dt} + cZ^2 - C(T) = -\frac{1}{ct^2} + c \left(\frac{1}{ct} + \sqrt{\frac{C(T)}{c}} \right)^2 - C(T) = \frac{2\sqrt{\frac{C(T)}{c}}}{t} \geq 0.$$

Then, for every $t \in (0, T)$, $Z(t)$ is a supersolution of (4.5). The comparison principle for parabolic equation and the comparison principle for ordinary differential equations give

$$\partial_x u_\varepsilon(t, x) \leq z(t) \leq Z(t) = \frac{1}{ct} + \sqrt{\frac{C(T)}{c}}, \quad t \in (0, T), \quad x \in \mathbb{R},$$

that is (4.3). Since for every $t \in (0, T)$ and $x, y \in \mathbb{R}$, $x \neq y$, thanks to (4.3),

$$\frac{u_\varepsilon(t, x) - u_\varepsilon(t, y)}{x - y} = \frac{1}{x - y} \int_y^x \partial_x u_\varepsilon(t, \xi) d\xi = \frac{1}{ct} + \sqrt{\frac{C(T)}{c}} \leq C(T) \left(\frac{1}{t} + 1 \right).$$

(3.40) gives (4.2). \square

Let us assume that there exist two bounded distributional solution u and v of (1.8), or (1.9), such that

$$(4.6) \quad \frac{u(t, x) - u(t, y)}{x - y} \leq C(T) \left(\frac{1}{t} + 1 \right), \quad \frac{v(t, x) - v(t, y)}{x - y} \leq C(T) \left(\frac{1}{t} + 1 \right),$$

for almost every $0 < t < T$, $x, y \in \mathbb{R}$, $x \neq y$, and some constant $C(T) > 0$. We want to prove that

$$(4.7) \quad u = v \quad \text{a.e. in } (0, T) \times \mathbb{R}.$$

Let $\phi \in C^\infty(\mathbb{R}^2)$ be a test function with compact support. Since u and v are distributional solutions of (1.8), or (1.9), we have that

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} (u \partial_t \phi + f(u) \partial_x \phi) dt dx + \gamma \int_0^\infty \int_{\mathbb{R}} \phi \left(\int_{-\infty}^x u(t, y) dy \right) dt dx + \int_{\mathbb{R}} \phi(0, x) u_0(x) dx &= 0, \\ \int_0^\infty \int_{\mathbb{R}} (v \partial_t \phi + f(v) \partial_x \phi) dt dx + \gamma \int_0^\infty \int_{\mathbb{R}} \phi \left(\int_{-\infty}^x v(t, y) dy \right) dt dx + \int_{\mathbb{R}} \phi(0, x) u_0(x) dx &= 0 \end{aligned}$$

and then

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} ((u - v) \partial_t \phi + (f(u) - f(v)) \partial_x \phi) dt dx \\ + \gamma \int_0^\infty \int_{\mathbb{R}} \phi \left(\int_{-\infty}^x (u(t, y) - v(t, y)) dy \right) dt dx = 0. \end{aligned}$$

Therefore, we have that

$$(4.8) \quad \int_0^\infty \int_{\mathbb{R}} w(\partial_t \phi + b \partial_x \phi) ds dx + \gamma \int_0^\infty \int_{\mathbb{R}} \phi \left(\int_{-\infty}^x w(t, y) dy \right) dt dx = 0,$$

where

$$(4.9) \quad w = u - v, \quad b(t, x) = \int_0^1 f'(\theta u(t, x) + (1 - \theta)v(t, x)) d\theta = \frac{f(u(t, x)) - f(v(t, x))}{u(t, x) - v(t, x)}.$$

Observe that, since

$$(4.10) \quad u, v \in L^\infty((0, T) \times \mathbb{R}),$$

we get

$$(4.11) \quad |f(u(t, x)) - f(v(t, x))| \leq C(T)|u(t, x) - v(t, x)|,$$

where

$$C(T) = \sup_{(0, T) \times \mathbb{R}} \left\{ |f'(u)| + |f'(v)| \right\}.$$

Therefore, for (4.2) and (4.11), on the function $b(t, x)$, we have the following estimates

$$(4.12) \quad \begin{aligned} \|b\|_{L^\infty((0, T) \times \mathbb{R})} &\leq C(T), \quad T > 0, \\ \frac{b(t, x) - b(t, y)}{x - y} &\leq C(T) \left(\frac{1}{t} + 1 \right), \quad x \neq y, \quad 0 < t < T. \end{aligned}$$

Now, let us consider the following set:

$$\Omega := \{(x, y) \in \mathbb{R}^2; \quad y \leq x\}.$$

Therefore,

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} \phi \left(\int_{-\infty}^x w(t, y) dy \right) ds dx &= \int_0^\infty \int_{\Omega} \phi w(t, y) dt dx dy \\ &= \int_0^\infty \int_{\mathbb{R}} w(t, y) \left(\int_y^\infty \phi dx \right) dt dy. \end{aligned}$$

Hence,

$$(4.13) \quad \int_0^\infty \int_{\mathbb{R}} \phi \left(\int_{-\infty}^x w(t, y) dy \right) dt dx = \int_0^\infty \int_{\mathbb{R}} \Phi(t, y) w(t, y) dt dy,$$

where

$$(4.14) \quad \Phi(t, y) = \int_y^\infty \phi(t, y) dy$$

It follows from (4.8) and (4.13) that

$$(4.15) \quad \int_0^\infty \int_{\mathbb{R}} w(\partial_t \phi + b \partial_x \phi + \gamma \Phi) ds dx = 0.$$

Fix $\psi \in C_c((0, \infty) \times \mathbb{R})$ and let $\tau > 0$ be such that

$$(4.16) \quad \text{supp}(\psi) \subset (0, \tau) \times \mathbb{R},$$

to have (4.7), we have to solve the following system:

$$(4.17) \quad \begin{cases} \partial_t \phi + b \partial_x \phi = \psi - \gamma \Phi, & (t, x) \in (0, \tau) \times \mathbb{R} \\ \partial_x \Phi = -\phi, & (t, x) \in (0, \tau) \times \mathbb{R} \\ \phi(\tau, x) = 0, & x \in \mathbb{R}. \end{cases}$$

We coin (4.17) the adjoint problem associated with (2.1).

The idea is to solve (4.17) and then pass from (4.15) to the following equation

$$(4.18) \quad \int_0^\infty \int_{\mathbb{R}} w \psi dt dx = 0.$$

Unfortunately, due to the low regularity of the coefficient b , we cannot solve directly (4.17). Hence, we regularize the first equation by smoothing the coefficient b by convolution and adding an artificial viscosity term.

The use of an adjoint problem to prove uniqueness is rather common in the context of first order conservation laws, see for example [7, 13, 23, 27, 29].

Let us consider $\{\rho_\varepsilon(t, x)\}_{\varepsilon>0}$ a sequence of standard mollifiers. Define

$$b_\varepsilon = b * \rho_\varepsilon, \quad \varepsilon > 0,$$

where $*$ denotes the convolution in both variables t and x .

Clearly, from (1.6), (4.9) and (4.12),

$$(4.19) \quad b_\varepsilon \rightarrow b, \quad \text{in } L^2((0, T) \times \mathbb{R}), \quad T > 0,$$

$$(4.20) \quad \|b_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T), \quad T, \varepsilon > 0,$$

$$(4.21) \quad \partial_x b_\varepsilon(t, x) \leq C(T) \left(\frac{1}{t} + 1 \right), \quad 0 < t < T, \quad x \in \mathbb{R}, \quad \varepsilon > 0.$$

Now, we approximate (4.17) in following way:

$$(4.22) \quad \begin{cases} \partial_t \phi_\varepsilon + b_\varepsilon \partial_x \phi_\varepsilon = \psi_\varepsilon - \gamma \Phi_\varepsilon - \varepsilon \partial_{xx}^2 \phi_\varepsilon, & (t, x) \in (0, \tau) \times \mathbb{R} \\ \partial_x \Phi_\varepsilon = -\phi_\varepsilon, & (t, x) \in (0, \tau) \times \mathbb{R} \\ \phi_\varepsilon(\tau, x) = 0, & x \in \mathbb{R}. \end{cases}$$

The existence of solutions for (4.22) is obtained considering the following system

$$\begin{cases} \partial_t \phi_{\varepsilon, \delta} + b_{\varepsilon, \delta} \partial_x \phi_{\varepsilon, \delta} = \psi_{\varepsilon, \delta} - \gamma \Phi_{\varepsilon, \delta} - \varepsilon \partial_{xx}^2 \phi_{\varepsilon, \delta}, & (t, x) \in (0, \tau) \times \mathbb{R} \\ -\delta \partial_{xx}^2 \Phi_{\varepsilon, \delta} + \partial_x \Phi_{\varepsilon, \delta} = -\phi_{\varepsilon, \delta}, & (t, x) \in (0, \tau) \times \mathbb{R} \\ \phi_{\varepsilon, \delta}(\tau, x) = 0, & x \in \mathbb{R}, \end{cases}$$

and sending $\delta \rightarrow 0$ (see Section 2).

Therefore, arguing as in Section 2, Theorem 2.1, we obtain

Lemma 4.1. *Let $\varepsilon > 0$ and suppose $\psi_\varepsilon \in C^\infty((0, \infty) \times \mathbb{R}) \cap C((0, \infty); H^2(\mathbb{R}))$ obeys (4.16). There exists a unique solution $\phi_\varepsilon \in C^\infty((0, \infty) \times \mathbb{R}) \cap C((0, \infty); H^\ell(\mathbb{R}))$, $\ell > 2$, to the terminal value problem (4.22).*

Since we feel more comfortable with initial value problems, we define

$$(4.23) \quad w_\varepsilon(t, x) = \phi_\varepsilon(\tau - t, x), \quad Q_\varepsilon(t, x) = \Phi_\varepsilon(\tau - t, x)$$

$$(4.24) \quad \beta_\varepsilon(t, x) = b_\varepsilon(\tau - t, x), \quad \tilde{\psi}_\varepsilon(\tau - t, x) = -\psi_\varepsilon(\tau - t, x),$$

for $(t, x) \in (0, \tau) \times \mathbb{R}$. Due to Lemma 4.1, w_ε is then the unique smooth solution of the initial value problem

$$(4.25) \quad \begin{cases} \partial_t w_\varepsilon - \beta_\varepsilon \partial_x w_\varepsilon = \tilde{\psi}_\varepsilon + \gamma Q_\varepsilon + \varepsilon \partial_{xx}^2 w_\varepsilon, & (t, x) \in (0, \tau) \times \mathbb{R} \\ \partial_x Q_\varepsilon = -w_\varepsilon, & (t, x) \in (0, \tau) \times \mathbb{R} \\ w_\varepsilon(0, x) = 0, & x \in \mathbb{R}. \end{cases}$$

Denoting with $C(\tau)$ the constants which depends on τ , thanks to (4.20), (4.21) and (4.24), we get

$$(4.26) \quad \|\beta_\varepsilon\|_{L^\infty((0, \tau) \times \mathbb{R})} \leq C(\tau), \quad \varepsilon > 0,$$

$$(4.27) \quad \partial_x \beta_\varepsilon(t, x) \leq C(\tau) \left(\frac{1}{\tau - t} + 1 \right), \quad (t, x) \in (0, \tau) \times \mathbb{R}, \quad \varepsilon > 0.$$

We prove our key estimates.

Lemma 4.2. *Let $\psi_\varepsilon \in C^\infty((0, \infty) \times \mathbb{R}) \cap C((0, \infty); H^2(\mathbb{R})) \cap L^\infty((0, \infty); H^2(\mathbb{R}))$ be a function satisfying (4.16). Then, using the notation introduced in (4.23) and (4.24), there exists a function $C(\tau) > 0$, independent on ε such that*

$$(4.28) \quad \begin{aligned} & \|w_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 2\varepsilon \int_0^t \|\partial_x w_\varepsilon(s, \cdot)\|_{H^1(\mathbb{R})}^2 ds \\ & \leq e^{C(\tau)\tau} \left(\frac{\tau}{\tau-t} \right)^{C(\tau)} \int_0^t \|\tilde{\psi}(s, \cdot)\|_{H^1(\mathbb{R})}^2 ds, \end{aligned}$$

for every $t \in (0, \tau)$. In particular, we have that

$$(4.29) \quad \|w_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \sqrt{2} \left(e^{C(\tau)\tau} \left(\frac{\tau}{\tau-t} \right)^{C(\tau)} \int_0^t \|\tilde{\psi}(s, \cdot)\|_{H^1(\mathbb{R})}^2 ds \right)^{\frac{1}{2}}.$$

Proof. We begin by proving that (4.28) holds true. Multiplying the first equation of (4.25) by v , an integration on \mathbb{R} gives

$$(4.30) \quad \frac{d}{dt} \int_{\mathbb{R}} w_\varepsilon^2 dx = 2 \int_{\mathbb{R}} \beta_\varepsilon w_\varepsilon \partial_x w_\varepsilon dx + 2 \int_{\mathbb{R}} w_\varepsilon \tilde{\psi} dx + 2\gamma \int_{\mathbb{R}} Q_\varepsilon w_\varepsilon dx + 2\varepsilon \int_{\mathbb{R}} w_\varepsilon \partial_{xx}^2 w_\varepsilon dx.$$

Thanks the second equation of (4.25), we have that

$$(4.31) \quad 2\gamma \int_{\mathbb{R}} Q_\varepsilon(t, x) w_\varepsilon(t, x) dx = -2\gamma \int_{\mathbb{R}} Q_\varepsilon(t, x) \partial_x Q_\varepsilon(t, x) dx = 0.$$

Therefore, it follows from (4.26), (4.30), (4.31) and the Young inequality that

$$(4.32) \quad \begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} w_\varepsilon^2 dx + 2\varepsilon \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 dx \\ & = 2 \int_{\mathbb{R}} \beta_\varepsilon w_\varepsilon \partial_x w_\varepsilon dx + 2 \int_{\mathbb{R}} w_\varepsilon \tilde{\psi} dx \\ & \leq 2 \int_{\mathbb{R}} |\beta_\varepsilon| |w_\varepsilon| |\partial_x w_\varepsilon| dx + 2 \int_{\mathbb{R}} |w_\varepsilon| |\tilde{\psi}| dx \\ & \leq 2C(\tau) \int_{\mathbb{R}} |w_\varepsilon| |\partial_x w_\varepsilon| dx + \int_{\mathbb{R}} w_\varepsilon^2 dx + \int_{\mathbb{R}} \tilde{\psi}^2 dx \\ & \leq C(\tau) \int_{\mathbb{R}} w_\varepsilon^2 dx + C(\tau) \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 dx + \int_{\mathbb{R}} \tilde{\psi}^2 dx. \end{aligned}$$

Differentiating with respect to x the first equation of (4.25), we get

$$(4.33) \quad \partial_{tx}^2 w_\varepsilon = \partial_x \beta_\varepsilon \partial_x w_\varepsilon + \beta_\varepsilon \partial_{xx}^2 w_\varepsilon + \partial_x \tilde{\psi} + \gamma \partial_x Q_\varepsilon + \varepsilon \partial_{xxx}^3 w_\varepsilon.$$

The second equation of (4.25) and (4.33) give

$$(4.34) \quad \partial_{tx}^2 w_\varepsilon = \partial_x \beta_\varepsilon \partial_x w_\varepsilon + \beta_\varepsilon \partial_{xx}^2 w_\varepsilon + \partial_x \tilde{\psi} - \gamma w_\varepsilon + \varepsilon \partial_{xxx}^3 w_\varepsilon$$

Multiplying (4.34) by $\partial_x w_\varepsilon$, we obtain that

$$\begin{aligned} \int_{\mathbb{R}} \partial_{tx}^2 w_\varepsilon \partial_x w_\varepsilon dx &= \int_{\mathbb{R}} \partial_x \beta_\varepsilon (\partial_x w_\varepsilon)^2 dx + \int_{\mathbb{R}} \beta_\varepsilon \partial_{xx}^2 w_\varepsilon \partial_x w_\varepsilon dx + \int_{\mathbb{R}} \partial_x \tilde{\psi} \partial_x w_\varepsilon dx \\ &\quad - \gamma \int_{\mathbb{R}} w_\varepsilon \partial_x w_\varepsilon dx + \varepsilon \int_{\mathbb{R}} \partial_{xxx}^3 w_\varepsilon \partial_x w_\varepsilon dx. \end{aligned}$$

Since,

$$\int_{\mathbb{R}} \partial_{tx}^2 w_\varepsilon \partial_x w_\varepsilon dx = \frac{d}{dt} \left(\frac{1}{2} \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 dx \right),$$

$$\begin{aligned} \int_{\mathbb{R}} \beta_\varepsilon \partial_{xx}^2 w_\varepsilon \partial_x w_\varepsilon dx &= -\frac{1}{2} \int_{\mathbb{R}} \partial_x \beta_\varepsilon (\partial_x w_\varepsilon)^2 dx, \\ -\gamma \int_{\mathbb{R}} w_\varepsilon \partial_x w_\varepsilon dx &= 0, \end{aligned}$$

due to (4.27) and the Young inequality, we have that

$$\begin{aligned} (4.35) \quad & \frac{d}{dt} \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 dx + 2\varepsilon \int_{\mathbb{R}} (\partial_{xx}^2 w_\varepsilon)^2 dx \\ &= \int_{\mathbb{R}} \partial_x \beta_\varepsilon (\partial_x w_\varepsilon)^2 dx + 2 \int_{\mathbb{R}} \partial_x \tilde{\psi} \partial_x w_\varepsilon dx \\ &\leq C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 dx + 2 \int_{\mathbb{R}} |\partial_x \tilde{\psi}| |\partial_x w_\varepsilon| dx \\ &\leq C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 dx + \int_{\mathbb{R}} (\partial_x \tilde{\psi})^2 dx + \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 dx \\ &\leq C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \int_{\mathbb{R}} (\partial_x w_\varepsilon)^2 dx + \int_{\mathbb{R}} (\partial_x \tilde{\psi})^2 dx. \end{aligned}$$

Adding (4.32) and (4.35), we obtain that

$$\begin{aligned} & \frac{d}{dt} \|w_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 2\varepsilon \|\partial_x w_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 \\ &\leq C(\tau) \|w_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(\tau) \|\partial_x w_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\quad + C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \|\partial_x w_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\tilde{\psi}(t, \cdot)\|_{H^1(\mathbb{R})}^2 \\ &\leq C(\tau) \|w_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 + C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \|\partial_x w_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \\ &\quad + \|\tilde{\psi}(t, \cdot)\|_{H^1(\mathbb{R})}^2 + C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \|w_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Therefore,

$$\begin{aligned} (4.36) \quad & \frac{d}{dt} \|w_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 2\varepsilon \|\partial_x w_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 \\ &\leq C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \|w_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + \|\tilde{\psi}(t, \cdot)\|_{H^1(\mathbb{R})}^2. \end{aligned}$$

Let $f(t)$ be a nonnegative, absolutely continuous function on $[a, b]$, satisfying for *a.e.* t the inequality

$$f'(t) + g(t) \leq k(t)f(t) + h(t),$$

where $k(t)$, $g(t)$, $h(t)$ are nonnegative functions on $[a, b]$. Then, the Gronwall inequality says that

$$f(t) + \int_a^b e^{\int_s^t k(s') ds'} g(s) ds \leq e^{\int_a^t k(s) ds} \left(f(a) + \int_a^t h(s) ds \right), \quad a \leq t \leq b.$$

For (4.36), $k(t) = C(\tau) \left(\frac{1}{\tau-t} + 1 \right)$ and thus $e^{\int_s^t k(s') ds'} = e^{C(\tau)(t-s)} \left(\frac{\tau-s}{\tau-t} \right)^{C(\tau)}$, so we obtain, keeping in mind that $\partial_x v(0, \cdot) = 0$,

$$(4.37) \quad \begin{aligned} & \|w_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 + 2\varepsilon \int_0^t e^{C(\tau)(t-s)} \left(\frac{\tau-s}{\tau-t} \right)^{C(\tau)} \|\partial_x w_\varepsilon(s, \cdot)\|_{H^1(\mathbb{R})}^2 ds \\ & \leq e^{C(\tau)(t-s)} \left(\frac{\tau}{\tau-t} \right)^{C(\tau)} \int_0^t \|\tilde{\psi}(s, \cdot)\|_{H^1(\mathbb{R})}^2 ds. \end{aligned}$$

Since $s \leq t$, then $\tau - s \geq \tau - t$. Therefore,

$$(4.38) \quad 1 \leq \frac{\tau-s}{\tau-t}.$$

Thus, (4.37) and (4.37) give (4.28).

Finally, we prove (4.29). Due to (4.28) and the Hölder inequality, we get

$$\begin{aligned} w_\varepsilon^2(t, x) &= 2 \int_{-\infty}^x w_\varepsilon(t, y) \partial_x w_\varepsilon(t, y) dy \leq 2 \int_{\mathbb{R}} |w_\varepsilon(t, y)| |\partial_x w_\varepsilon(t, y)| dx \\ &\leq 2 \|w_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})} \leq e^{C(\tau)\tau} \left(\frac{\tau}{\tau-t} \right)^{C(\tau)} \int_0^t \|\tilde{\psi}(s, \cdot)\|_{H^1(\mathbb{R})}^2 ds. \end{aligned}$$

Therefore,

$$(4.39) \quad |w_\varepsilon(t, x)| \leq \sqrt{2} \left(e^{C(\tau)\tau} \left(\frac{\tau}{\tau-t} \right)^{C(\tau)} \int_0^t \|\tilde{\psi}(s, \cdot)\|_{H^1(\mathbb{R})}^2 ds \right)^{\frac{1}{2}}.$$

(4.29) follows from (4.39). \square

Lemma 4.3. *Let $\psi_\varepsilon \in C^\infty((0, \infty) \times \mathbb{R}) \cap C((0, \infty); H^2(\mathbb{R})) \cap L^\infty((0, \infty); H^2(\mathbb{R}))$ be a function satisfying (4.16). Then, using the notation introduced in (4.23) and (4.24), there exists a function $C(\tau) > 0$, independent on ε such that*

$$(4.40) \quad \|\partial_x w_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C(\tau) \tau \left(\frac{\tau}{\tau-t} \right)^{C(\tau)} e^{C(\tau)\tau}.$$

Proof. Let $p \in \mathbb{N} \setminus \{0\}$ be even. Thanks to (4.34),

$$\begin{aligned} \frac{d}{dt} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^p &= \frac{d}{dt} \int_{\mathbb{R}} (\partial_x w_\varepsilon)^p dx = p \int_{\mathbb{R}} (\partial_x w_\varepsilon)^{p-1} \partial_{tx}^2 w_\varepsilon dx \\ &= p \int_{\mathbb{R}} \partial_x \beta_\varepsilon (\partial_x w_\varepsilon)^p dx + p \int_{\mathbb{R}} \beta_\varepsilon \partial_{xx}^2 w_\varepsilon (\partial_x w_\varepsilon)^{p-1} dx + p \int_{\mathbb{R}} \partial_x \tilde{\psi} (\partial_x w_\varepsilon)^{p-1} dx \\ &\quad - p\gamma \int_{\mathbb{R}} w_\varepsilon (\partial_x w_\varepsilon)^{p-1} dx + \varepsilon p \int_{\mathbb{R}} \partial_{xxx}^3 w_\varepsilon (\partial_x w_\varepsilon)^{p-1} dx \\ &= (p-1) \int_{\mathbb{R}} \partial_x \beta_\varepsilon (\partial_x w_\varepsilon)^p dx + p \int_{\mathbb{R}} \partial_x \tilde{\psi} (\partial_x w_\varepsilon)^{p-1} dx - p\gamma \int_{\mathbb{R}} w_\varepsilon (\partial_x w_\varepsilon)^{p-1} dx \\ &\quad - p(p-1)\varepsilon \int_{\mathbb{R}} (\partial_x w_\varepsilon)^{p-2} (\partial_{xx}^2 w_\varepsilon)^2 dx, \end{aligned}$$

that is,

$$\begin{aligned} \frac{d}{dt} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^p &\leq (p-1) \int_{\mathbb{R}} \partial_x \beta_\varepsilon (\partial_x w_\varepsilon)^p dx \\ &\quad + p \int_{\mathbb{R}} \partial_x \tilde{\psi} (\partial_x w_\varepsilon)^{p-1} dx - p\gamma \int_{\mathbb{R}} w_\varepsilon (\partial_x w_\varepsilon)^{p-1} dx. \end{aligned}$$

Due to (4.27) and the Hölder inequality, we get

$$\begin{aligned}
\frac{d}{dt} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^p &\leq (p-1)C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^p \\
&\quad + p \int_{\mathbb{R}} |\partial_x \tilde{\psi}| |(\partial_x w_\varepsilon)^{p-1}| dx + p\gamma \int_{\mathbb{R}} |w_\varepsilon| |(\partial_x w_\varepsilon)^{p-1}| dx \\
&\leq (p-1)C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^p \\
&\quad + p \left\| \partial_x \tilde{\psi}(t, \cdot) \right\|_{L^p(\mathbb{R})} \left\| (\partial_x w_\varepsilon(t, \cdot))^{p-1} \right\|_{L^{\frac{p}{p-1}}(\mathbb{R})} \\
&\quad + p\gamma \|w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} \left\| (\partial_x w_\varepsilon(t, \cdot))^{p-1} \right\|_{L^{\frac{p}{p-1}}(\mathbb{R})}.
\end{aligned}$$

Since

$$\left\| \partial_x \tilde{\psi}(t, \cdot) \right\|_{L^p(\mathbb{R})} \leq \alpha_1,$$

where α_1 is a positive constant which does not depend on ε , we have that

$$\begin{aligned}
(4.41) \quad \frac{d}{dt} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^p &\leq (p-1)C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^p \\
&\quad + p\alpha_1 \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^{p-1} \\
&\quad + p\gamma \|w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^{p-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
p \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^{p-1} \frac{d}{dt} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} &\leq (p-1)C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^p \\
&\quad + p\alpha_1 \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^{p-1} + p\gamma \|w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}^{p-1},
\end{aligned}$$

that is

$$\begin{aligned}
(4.42) \quad \frac{d}{dt} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} &\leq \frac{(p-1)}{p} C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} \\
&\quad + \alpha_1 + \gamma \|w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})}.
\end{aligned}$$

Due to (4.28) and (4.29),

$$\begin{aligned}
\int_{\mathbb{R}} |w_\varepsilon(t, x)|^p dx &= \int_{\mathbb{R}} |w_\varepsilon(t, x)|^{p-2} w_\varepsilon^2(t, x) dx \\
&\leq \|w_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})}^{p-2} \|w_\varepsilon(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq (F_1(\tau, t, \tilde{\psi}))^{p-2} (F_2(\tau, t, \tilde{\psi})),
\end{aligned}$$

where

$$\begin{aligned}
(4.43) \quad F_1(\tau, t, \tilde{\psi}) &= \sqrt{2} \left(e^{C(\tau)\tau} \left(\frac{\tau}{\tau-t} \right)^{C(\tau)} \int_0^t \left\| \tilde{\psi}(s, \cdot) \right\|_{H^1(\mathbb{R})}^2 ds \right)^{\frac{1}{2}}, \\
F_2(\tau, t, \tilde{\psi}) &= e^{C(\tau)\tau} \left(\frac{\tau}{\tau-t} \right)^{C(\tau)} \int_0^t \left\| \tilde{\psi}(s, \cdot) \right\|_{H^1(\mathbb{R})}^2 ds.
\end{aligned}$$

Thus,

$$(4.44) \quad \|w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} \leq (F_1(\tau, t, \tilde{\psi}))^{\frac{p-2}{p}} (F_2(\tau, t, \tilde{\psi}))^{\frac{1}{p}}.$$

Hence, (4.42) and (4.44) give

$$(4.45) \quad \begin{aligned} \frac{d}{dt} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} &\leq \frac{(p-1)}{p} C(\tau) \left(\frac{1}{\tau-t} + 1 \right) \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} \\ &\quad + \alpha_1 + \gamma (F_1(\tau, t, \tilde{\psi}))^{\frac{p-2}{p}} (F_2(\tau, t, \tilde{\psi}))^{\frac{1}{p}}. \end{aligned}$$

Keeping in mind that $\partial_x v(0, \cdot) = 0$, the Gronwall Lemma gives

$$\begin{aligned} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} &\leq \alpha_1 e^{\frac{p-1}{p} C(\tau) (\log(\frac{\tau}{\tau-t}) + t)} \int_0^t e^{-\frac{p-1}{p} C(\tau) (\log(\frac{\tau}{\tau-s}) + s)} ds \\ &\quad + \gamma e^{\frac{p-1}{p} C(\tau) (\log(\frac{\tau}{\tau-t}) + t)} \int_0^t e^{-\frac{p-1}{p} C(\tau) (\log(\frac{\tau}{\tau-s}) + s)} \\ &\quad \cdot (F_1(\tau, s, \tilde{\psi}))^{\frac{p-2}{p}} (F_2(\tau, s, \tilde{\psi}))^{\frac{1}{p}} ds \\ &\leq \alpha_1 \left(\frac{\tau}{\tau-t} \right)^{C(\tau) \frac{p-1}{p}} e^{\frac{p-1}{p} C(\tau) t} \int_0^t e^{-\frac{p-1}{p} C(\tau) s} \left(\frac{\tau-s}{\tau} \right)^{\frac{p-1}{p} C(\tau)} ds \\ &\quad + \gamma \left(\frac{\tau}{\tau-t} \right)^{C(\tau) \frac{p-1}{p}} e^{\frac{p-1}{p} C(\tau) t} \int_0^t e^{-\frac{p-1}{p} C(\tau) s} \left(\frac{\tau-s}{\tau} \right)^{\frac{p-1}{p} C(\tau)} \\ &\quad \cdot (F_1(\tau, s, \tilde{\psi}))^{\frac{p-2}{p}} (F_2(\tau, s, \tilde{\psi}))^{\frac{1}{p}} ds \\ &\leq \alpha_1 \left(\frac{\tau}{\tau-t} \right)^{C(\tau) \frac{p-1}{p}} e^{\frac{p-1}{p} C(\tau) \tau} \tau + \gamma \left(\frac{\tau}{\tau-t} \right)^{C(\tau) \frac{p-1}{p}} e^{\frac{p-1}{p} C(\tau) \tau} \\ &\quad \cdot \int_0^t \left(\frac{\tau-s}{\tau} \right)^{\frac{p-1}{p} C(\tau)} (F_1(\tau, s, \tilde{\psi}))^{\frac{p-2}{p}} (F_2(\tau, s, \tilde{\psi}))^{\frac{1}{p}} ds. \end{aligned}$$

We observe that, for (4.43),

$$\begin{aligned} &\int_0^t \left(\frac{\tau-s}{\tau} \right)^{\frac{p-1}{p} C(\tau)} (F_1(\tau, s, \tilde{\psi}))^{\frac{p-2}{p}} (F_2(\tau, s, \tilde{\psi}))^{\frac{1}{p}} ds \\ &\leq 2^{\frac{p-2}{2p}} \alpha_2^{\frac{1}{2}} \int_0^t e^{C(\tau) \tau \frac{p-2}{2p}} e^{C(\tau) \tau \frac{1}{p}} \left(\frac{\tau-s}{\tau} \right)^{\frac{p-1}{p} C(\tau)} \left(\frac{\tau}{\tau-s} \right)^{C(\tau) \frac{p-2}{2p}} \\ &\quad \cdot \left(\frac{\tau}{\tau-s} \right)^{C(\tau) \frac{1}{p}} s^{\frac{1}{2}} ds \\ &\leq 2^{\frac{p-2}{2p}} \alpha_2^{\frac{1}{2}} t^{\frac{1}{2}} e^{\frac{C(\tau) \tau}{2}} \int_0^t \left(\frac{\tau-s}{\tau} \right)^{\frac{p-1}{p} C(\tau)} \left(\frac{\tau-s}{\tau} \right)^{-C(\tau) \frac{p-2}{2p}} \\ &\quad \cdot \left(\frac{\tau-s}{\tau} \right)^{-C(\tau) \frac{1}{p}} ds \\ &\leq 2^{\frac{p-2}{2p}} \alpha_2^{\frac{1}{2}} \tau^{\frac{1}{2}} e^{\frac{C(\tau) \tau}{2}} \int_0^t \left(\frac{\tau-s}{\tau} \right)^{\frac{p-2}{2p} C(\tau)} ds \\ &\leq 2^{\frac{p-2}{2p}} \alpha_2^{\frac{1}{2}} \tau^{\frac{1}{2}} e^{\frac{C(\tau) \tau}{2}} \int_0^t ds \leq 2^{\frac{p-2}{2p}} \alpha_2^{\frac{1}{2}} \tau^{\frac{3}{2}} e^{\frac{C(\tau) \tau}{2}}, \end{aligned}$$

where α_2 is a positive constant independent on ε such that

$$\|\tilde{\psi}(s, \cdot)\|_{H^1(\mathbb{R})}^2 \leq \alpha_2.$$

Hence,

$$\begin{aligned} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} &\leq \alpha_1 \left(\frac{\tau}{\tau-t} \right)^{C(\tau) \frac{p-1}{p}} e^{\frac{p-1}{p} C(\tau) \tau} \tau \\ &\quad + \gamma 2^{\frac{p-2}{2p}} \alpha_2^{\frac{1}{2}} \tau^{\frac{3}{2}} \left(\frac{\tau}{\tau-t} \right)^{C(\tau) \frac{p-1}{p}} e^{\frac{p-1}{p} C(\tau) \tau} e^{\frac{C(\tau) \tau}{2}} \\ &\leq \left(\alpha_1 + \gamma \alpha_2^{\frac{1}{2}} 2^{\frac{p-2}{2p}} \tau^{\frac{1}{2}} e^{\frac{C(\tau) \tau}{2}} \right) \tau \left(\frac{\tau}{\tau-t} \right)^{C(\tau) \frac{p-1}{p}} e^{\frac{p-1}{p} C(\tau) \tau}. \end{aligned}$$

Sending $p \rightarrow \infty$, we have

$$\begin{aligned} \|\partial_x w_\varepsilon(t, \cdot)\|_{L^p(\mathbb{R})} &\leq \left(\alpha_1 + 2\gamma \alpha_2^{\frac{1}{2}} \tau^{\frac{1}{2}} e^{\frac{C(\tau) \tau}{2}} \right) \tau \left(\frac{\tau}{\tau-t} \right)^{C(\tau)} e^{C(\tau) \tau} \\ &\leq C(\tau) \tau \left(\frac{\tau}{\tau-t} \right)^{C(\tau)} e^{C(\tau) \tau}, \end{aligned}$$

which gives (4.40). \square

Coming back to the terminal value problem, the previous results for the initial value problem translate into the following ones for (4.22):

Corollary 4.1. *Let $\psi_\varepsilon \in C^\infty((0, \infty) \times \mathbb{R}) \cap C((0, \infty); H^2(\mathbb{R})) \cap L^\infty((0, \infty); H^2(\mathbb{R}))$ be a function satisfying (4.16). Then for each $\varepsilon > 0$ and $t \in (0, \tau)$*

$$\begin{aligned} (4.46) \quad \|\phi_\varepsilon(t, \cdot)\|_{H^1(\mathbb{R})}^2 &+ 2\varepsilon \int_t^\tau \|\partial_x \phi_\varepsilon(s, \cdot)\|_{H^1(\mathbb{R})}^2 ds \\ &\leq e^{C(\tau) \tau} \left(\frac{\tau}{t} \right)^{C(\tau)} \int_t^\tau \|\psi_\varepsilon(s, \cdot)\|_{H^1(\mathbb{R})}^2 ds, \end{aligned}$$

$$(4.47) \quad \|\phi_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \sqrt{2} \left(e^{C(\tau) \tau} \left(\frac{\tau}{t} \right)^{C(\tau)} \int_t^\tau \|\psi_\varepsilon(s, \cdot)\|_{H^1(\mathbb{R})}^2 ds \right)^{\frac{1}{2}},$$

$$(4.48) \quad \|\partial_x \phi_\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C(\tau) \tau \left(\frac{\tau}{t} \right)^{C(\tau)} e^{C(\tau) \tau}.$$

Although we will not use this fact directly, an interesting consequence of the previous estimates is the existence of a solution of (4.17)

Theorem 4.2. *Fix any $0 < \delta < \tau$. Then there exists at least one distributional solution $(\phi, \Phi) \in L^\infty((\delta, \tau); W^{1,\infty}(\mathbb{R}) \cap H^1(\mathbb{R})) \times L^\infty((\delta, \tau); W^{2,\infty}(\mathbb{R}) \cap H^2(\mathbb{R}))$ to the terminal value problem (4.17).*

Proof. For each fixed $\varepsilon > 0$, let (ϕ, Φ) denote the solution of (4.22). Due to the second equation of (4.22) and Corollary 4.1,

$$(4.49) \quad \begin{aligned} \{\phi_\varepsilon\}_{\varepsilon>0} &\text{ is bounded in } L^\infty((\delta, \tau); W^{1,\infty}(\mathbb{R}) \cap H^1(\mathbb{R})), \quad \text{for } \delta \in (0, \tau), \\ \{\Phi_\varepsilon\}_{\varepsilon>0} &\text{ is bounded in } L^\infty((\delta, \tau); W^{2,\infty}(\mathbb{R}) \cap H^2(\mathbb{R})), \quad \text{for } \delta \in (0, \tau). \end{aligned}$$

Then, there exist

$$\phi \in L^\infty((\delta, \tau); W^{1,\infty}(\mathbb{R}) \cap H^1(\mathbb{R})), \quad \Phi \in L^\infty((\delta, \tau); W^{2,\infty}(\mathbb{R}) \cap H^2(\mathbb{R})), \quad 0 < \delta < \tau,$$

and $\{\varepsilon_k\}_{k \in \mathbb{N}}$, $\varepsilon_k \rightarrow 0$, such that

$$(4.50) \quad \begin{aligned} \phi_{\varepsilon_k} &\rightharpoonup \phi \quad \text{weakly in } L^p((\delta, \tau); W^{1,q}(\mathbb{R})), \quad \text{for } \delta \in (0, \tau), \quad 1 \leq p < \infty, \quad 2 \leq q < \infty, \\ \Phi_{\varepsilon_k} &\rightharpoonup \Phi \quad \text{weakly in } L^p((\delta, \tau); W^{2,q}(\mathbb{R})), \quad \text{for } \delta \in (0, \tau), \quad 1 \leq p < \infty, \quad 2 \leq q < \infty. \end{aligned}$$

It remains to verify that the limit pair (ϕ, Φ) is a solution of (4.17) in the sense of distributions. Fix any $\phi \in C_c^\infty((0, \tau) \times \mathbb{R})$. We need to show that

$$(4.51) \quad \int_0^\tau \int_{\mathbb{R}} \phi b_{\varepsilon_k} \partial_x \phi_{\varepsilon_k} dt dx \rightarrow \int_0^\tau \int_{\mathbb{R}} \phi b \partial_x \phi dt dx.$$

Observe that

$$(4.52) \quad \begin{aligned} \int_0^\tau \int_{\mathbb{R}} \phi (b_{\varepsilon_k} \partial_x \phi_{\varepsilon_k} - b \partial_x \phi) dt dx &= \int_0^\tau \int_{\mathbb{R}} \phi (b_{\varepsilon_k} - b) \partial_x \phi_{\varepsilon_k} dt dx \\ &\quad + \int_0^\tau \int_{\mathbb{R}} \phi b (\partial_x \phi_{\varepsilon_k} - \partial_x \phi) dt dx. \end{aligned}$$

Since ϕ has compact support in $(0, \tau) \times \mathbb{R}$, there exists $\delta > 0$ such that $\text{supp}(\phi) \subset (\delta, \tau) \times \mathbb{R}$. Therefore, we can employ (4.19) and (4.49) to obtain

$$(4.53) \quad \begin{aligned} \int_0^\tau \int_{\mathbb{R}} \phi (b_{\varepsilon_k} - b) \partial_x \psi_{\varepsilon_k} dt dx &= \int_\delta^\tau \int_{\mathbb{R}} \phi (b_{\varepsilon_k} - b) \partial_x \psi_{\varepsilon_k} dt dx \\ &\leq \|b_{\varepsilon_k} - b\|_{L^2((0, \tau) \times \mathbb{R})} \|\phi\|_{L^\infty((0, \tau) \times \mathbb{R})} \|\partial_x \psi_{\varepsilon_k}\|_{L^2((\delta, \tau) \times \mathbb{R})} \rightarrow 0. \end{aligned}$$

Since $\phi b \in L^2((0, \tau) \times \mathbb{R})$ and $\text{supp}(\phi b) \subset (\delta, \tau) \times \mathbb{R}$, it follows from (4.50) that

$$(4.54) \quad \int_0^\tau \int_{\mathbb{R}} \phi b (\partial_x \phi_{\varepsilon_k} - \partial_x \phi) dt dx \rightarrow 0.$$

(4.52), (4.53) and (4.54) give (4.51), and the proof is completed. \square

Now, we are ready for the proof of Theorem 1.1.

Proof of Theorem 1.1. In Section 3, it has been proved the existence of an entropy solution of (1.8), or (1.9). Moreover, for Theorem 4.1, we have that *i*) implies *ii*).

Let us show that *ii*) implies *i*). It is sufficient to prove that there exists a unique weak solution of (1.8), or (1.9), that verifies (1.12). Let us suppose that u and v are two weak solution of (1.8), or (1.9). We have to prove that (4.7) holds true.

We begin by fixed a test function $\psi \in C_c^\infty((0, \infty) \times \mathbb{R})$. Let $0 < \tau_0 < \tau_1$ be such that

$$(4.55) \quad \text{supp}(\psi) \subset (\tau_0, \tau_1) \times \mathbb{R}.$$

From Lemma 4.1, for each $\varepsilon > 0$ there exists a unique $\tilde{\phi}_\varepsilon \in C^\infty((0, \infty) \times \mathbb{R}) \cap C((0, \infty); H^\ell(\mathbb{R}))$, $\ell > 2$, solving (4.22). Let $\{\phi_\varepsilon\}_\varepsilon \subset C_c^\infty((0, \tau_1) \times \mathbb{R})$ be such that

$$(4.56) \quad \varepsilon |\text{supp}(\phi_\varepsilon)| \rightarrow 0,$$

$$(4.57) \quad \tilde{\phi}_\varepsilon - \phi_\varepsilon \rightarrow 0 \quad \text{strongly in} \quad \begin{cases} L^1((0, \infty); W^{2,1}(\mathbb{R})) \cap W^{1,1}((0, \infty) \times \mathbb{R}) \\ \cap W^{1,\infty}((0, \infty); H^1(\mathbb{R})) \cap L^\infty((0, \infty); H^\ell(\mathbb{R})), \end{cases}$$

with $\ell > 2$, and define the family $\{\psi_\varepsilon\}_\varepsilon$ as follows

$$(4.58) \quad \psi_\varepsilon = \partial_t \phi_\varepsilon + b_\varepsilon \partial_x \phi_\varepsilon + \gamma \Phi_\varepsilon + \varepsilon \partial_{xx}^2 \phi_\varepsilon, \quad \varepsilon > 0.$$

Clearly,

$$\psi_\varepsilon \in C^\infty((0, \infty) \times \mathbb{R}) \cap C((0, \infty); H^2(\mathbb{R})) \quad \varepsilon > 0,$$

and, due to (4.19), (4.20) and (4.57),

$$(4.59) \quad \psi_\varepsilon \rightarrow \psi \quad \text{strongly in} \quad L^1((0, \infty) \times \mathbb{R}) \cap L^\infty((0, \infty); H^2(\mathbb{R})).$$

In particular, ϕ_ε and ψ_ε satisfy the two equations (see (4.22) and (4.58))

$$(4.60) \quad \partial_t \phi_\varepsilon + b_\varepsilon \partial_x \phi_\varepsilon + \gamma \Phi_\varepsilon = \psi_\varepsilon - \varepsilon \partial_{xx}^2 \phi_\varepsilon, \quad \partial_x \Phi_\varepsilon = -\phi_\varepsilon.$$

Hence, using (4.55) and (4.60),

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}} \omega \psi dt dx &= \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega \psi dt dx \\
&= \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega \psi_\varepsilon dt dx + \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega (\psi - \psi_\varepsilon) dt dx \\
&= \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega (\partial_t \phi_\varepsilon + b_\varepsilon \partial_x \phi_\varepsilon + \gamma \Phi_\varepsilon + \varepsilon \partial_{xx}^2 \phi_\varepsilon) dt dx \\
&\quad + \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega (\psi - \psi_\varepsilon) dt dx \\
&= \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega (\partial_t \phi_\varepsilon + b \partial_x \phi_\varepsilon + \gamma \Phi_\varepsilon) dt dx \\
&\quad + \varepsilon \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega \partial_{xx}^2 \phi_\varepsilon dt dx + \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega (b_\varepsilon - b) \partial_x \phi_\varepsilon dt dx \\
&\quad + \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega (\psi - \psi_\varepsilon) dt dx.
\end{aligned} \tag{4.61}$$

Using the fact that $\phi_\varepsilon \in C_c^\infty((0, \infty) \times \mathbb{R})$ and (4.15), we have that

$$\int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega (\partial_t \phi_\varepsilon + b \partial_x \phi_\varepsilon + \gamma \Phi_\varepsilon) dt dx = 0. \tag{4.62}$$

Employing (4.10), (4.46), (4.56) and the Hölder inequality

$$\begin{aligned}
\left| \varepsilon \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega \partial_{xx}^2 \phi dt dx \right| &\leq \varepsilon \|\omega\|_{L^\infty((\tau_0, \tau_1) \times \mathbb{R})} \|\partial_{xx}^2 \phi_\varepsilon\|_{L^1((\tau_0, \tau_1) \times \mathbb{R})} \\
&\leq \varepsilon \|\omega\|_{L^\infty((\tau_0, \tau_1) \times \mathbb{R})} \sqrt{|\text{supp}(\phi_\varepsilon)|} \|\partial_{xx}^2 \phi_\varepsilon\|_{L^2((\tau_0, \tau_1) \times \mathbb{R})} \\
&\leq \left(\varepsilon \frac{|\text{supp}(\phi_\varepsilon)|}{2} \right)^{\frac{1}{2}} \|\omega\|_{L^\infty((\tau_0, \tau_1) \times \mathbb{R})} \\
&\quad \cdot e^{\frac{C(\tau)\tau}{2}} \left(\frac{\tau_1}{\tau_0} \right)^{\frac{C(\tau)}{2}} \left(\int_{\tau_0}^{\tau_1} \|\psi_\varepsilon(s, \cdot)\|_{H^1(\mathbb{R})}^2 ds \right)^{\frac{1}{2}} \rightarrow 0.
\end{aligned} \tag{4.63}$$

It follows from (4.10), (4.19), (4.46) and the Hölder inequality that

$$\begin{aligned}
\left| \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega (b - b_\varepsilon) \partial_x \phi_\varepsilon dt dx \right| &\leq \|\omega\|_{L^\infty((\tau_0, \tau_1) \times \mathbb{R})} \|b - b_\varepsilon\|_{L^2((\tau_1, \tau_0) \times \mathbb{R})} \|\phi_\varepsilon\|_{L^2((\tau_1, \tau_0) \times \mathbb{R})} \\
&\leq \|\omega\|_{L^\infty((\tau_0, \tau_1) \times \mathbb{R})} \sqrt{\tau_1 - \tau_0} \|b - b_\varepsilon\|_{L^2((\tau_1, \tau_0) \times \mathbb{R})} \\
&\quad \cdot \left(\varepsilon \frac{|\text{supp}(\phi_\varepsilon)|}{2} \right)^{\frac{1}{2}} \|\omega\|_{L^\infty((\tau_0, \tau_1) \times \mathbb{R})} e^{\frac{C(\tau)\tau}{2}} \\
&\quad \cdot \left(\frac{\tau_1}{\tau_0} \right)^{\frac{C(\tau)}{2}} \left(\int_{\tau_0}^{\tau_1} \|\psi_\varepsilon(s, \cdot)\|_{H^1(\mathbb{R})}^2 ds \right)^{\frac{1}{2}} \rightarrow 0.
\end{aligned} \tag{4.64}$$

Due to (4.10) and (4.59), we get

$$\left| \int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega (\psi - \psi_\varepsilon) dt dx \right| \leq \|\omega\|_{L^\infty((\tau_0, \tau_1) \times \mathbb{R})} \|\psi - \psi_\varepsilon\|_{L^1((0, \infty) \times \mathbb{R})} \rightarrow 0. \tag{4.65}$$

Summarizing, using (4.62), (4.63), (4.64) and (4.65) in (4.61) yields

$$\int_{\tau_0}^{\tau_1} \int_{\mathbb{R}} \omega \psi dt dx = 0.$$

Due to the freedom in the choice of ψ , this implies (4.7), and the proof is completed. \square

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